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Quasi-$*$ structure on q-Poincaré algebras

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Abstract

We use braided groups to introduce a theory of $*$-structures on general inhomogeneous quantum groups, which we formulate as *quasi-$*$* Hopf algebras. This allows the construction of the tensor product of unitary representations up to a quantum cocycle isomorphism, which is a novel feature of the inhomogeneous case. Examples include q-Poincaré quantum group enveloping algebras in R-matrix form appropriate to the previous q-Euclidean and q-Minkowski space–time algebras $R_{21}\boldsymbol{x}_1\boldsymbol{x}_2 = \boldsymbol{x}_2\boldsymbol{x}_1 R$ and $R_{21}\boldsymbol{u}_1 R\boldsymbol{u}_2 = \boldsymbol{u}_2 R_{21}\boldsymbol{u}_1 R$. We obtain unitarity of the fundamental differential representations. We further show that the Euclidean and Minkowski–Poincaré quantum groups are twisting equivalent by another quantum cocycle.

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1. Introduction

As well as specific roles in physical systems, quantum groups in recent years have motivated a quite general and systematic development of a kind of q-deformed geometry. The basic algebraic ingredients are quite well-understood by now, at least for the geometry of q-deformed compact groups (typically quantum groups [1,2]) and q-deformed linear spaces (typically the more novel braided groups introduced by the author [3]). Not understood, however, is the full story regarding the role of the $*$-structure or complex conjugation in this q-geometry. This is obviously important for contact with physics, where q-deformed field theories are expected to be of interest either as providing a regularisation of infinities

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as poles at $q = 1$ [4] or as effective theories modelling the feedback of quantum or other effects on geometry at the Planck scale [5,6]. An understanding of the $*$-structure is needed also for better contact with other C^*-algebra approaches to noncommutative geometry [7].

The present work follows on from a previous one [8] where we studied the $*$-structures on linear braided groups. Now we combine the considerations there with a previous general construction [9] for inhomogeneous quantum groups in order to develop a theory of $*$-structures on these. Many authors have considered q-Poincaré and other inhomogeneous quantum groups as a key step for q-deformed physics but found that they do not (in the examples that concern us here) have $*$-structures obeying the usual axioms [10] of a Hopf $*$-algebra. In physical terms it means that the 'unitary' ($*$-preserving) representations of these inhomogeneous quantum groups are not closed under tensor product. This is problematic and tells us that we need a more radical q-deformation of the concept of 'unitarity' as well. We propose in the present paper a solution to this long-standing problem. It was announced briefly at the end of [11] and is developed now in detail. In physical terms, we will see that quantum deformation introduces a kind of 'anomaly' in the sense of a cocycle governing the breakdown of unitarity. Actually, something like this is to be expected because if we view q-deformation as a regularisation scheme for field theory, we do have to recover anomalies in the cases where they exist. For example, unlike dimensional regularisation there is no problem with the ϵ tensor and we have to expect a problem in different quarter to generate the $U(1)$ axial anomaly.

Namely, we introduce the notion if a *quasi-$*$* Hopf algebra, which is a Hopf algebra where the algebra part is a $*$-algebra and the coalgebra Δ, ϵ obeys

$$(*\otimes *)\circ\Delta\circ * = \mathcal{R}^{-1}(\tau\circ\Delta)\mathcal{R}, \qquad \overline{\epsilon(\ )} = \epsilon\circ *, \qquad \mathcal{R}^{*\otimes *} = \mathcal{R}_{21},$$
$$(\mathrm{id}\otimes\Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \qquad (\Delta\otimes\mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$$

for some element $\mathcal{R}$ in the tensor square. We use the usual notations as in [12,13] or the text [14]. The above axioms are a generalisation both of the usual axioms of a Hopf $*$-algebra and of Drinfeld's axioms of a quasitriangular Hopf algebra. They reduce to one iff they reduce to the other, in which case they reduce to a quasitriangular Hopf $*$-algebra of real type. This formulation covers the examples that interest us when q is real: one can also consider a different framework suitable for $\mathcal{R}^{*\otimes *} = \mathcal{R}^{-1}$ though we do not do so here explicitly. We arrive at these axioms in Section 2, where we study in detail the abstract construction of inhomogeneous quantum groups from the 'braided geometry' approach based on a process of *bosonisation* introduced by the author in [15]. This associated to any braided group (in our case the linear or 'momentum' sector of the inhomogeneous quantum group) an ordinary quantum group by a certain semidirect product construction. We also study in Section 2 how the bosonisation changes under twisting by a quantum cocycle, cf. the ideas of Drinfeld in [16].

In the 'braided approach' to q-deformed geometry we begin with braided group deformations of $\mathbb{R}^n$ as the basic objects. The braiding (and q) enters in a way that is conceptually different from other approaches to non-commutative geometry, namely as *braid statistics* with which we explicitly endow the co-ordinates of the q-deformed $\mathbb{R}^n$. This is the key difference between the more familiar quantum groups (which are bosonic objects) and the new braided groups. For a full introduction to the latter we refer to [17–19] as well as the orginal works [3,20], etc. There is a solid theory of braided matrices, braided linear algebra, braided addition, braided differentiation, 'Poincaré' quantum groups, differential forms, epsilon tensors and integration developed in a series of papers [3,9,21–24]. There are also natural candidates for q-Minkowski [25–27] and q-Euclidean [28] space–time algebras within this programme, making contact too with earlier pioneering work in [29–31] where the same space–time algebras were proposed directly by other means. The braided approach extended the latter works and put them, moreover, into a general R-matrix form as part of uniform theory of braided spaces. The present work on quasi-$*$ Hopf algebras applies to all the inhomogeneous quantum groups in this approach [9] for which suitable reality properties are met. This includes the q-Euclidean group of motions $\mathbb{R}_q^n \rtimes U_q(so_n)$ in any dimension, and also the four-dimensional Minkowski version. These and other examples are described in detail in Section 3. We also show that the inhomogeneous quantum groups for the four-dimensional Euclidean and Minkowski cases are related by twisting, extending the 'quantum wick rotation' in [28].

In Section 4 we look at the representation theory of quasi-$*$ Hopf algebras and inhomogeneous quantum groups from the point of view of our braided approach. This approach brings out the deeper meaning of the role of $*$ in braided geometry as a combined conjugation-braid reversal symmetry of our constructions. The seeds of this idea are already implicit in the earliest works on quantum groups, where the conjugate $l^{\pm}$ generators of a quantum enveloping algebra [32] are associated to a universal R-matrix and its inverse–transpose, respectively. In braided geometry all our constructions are done by means of braid and tangle diagrams representing the 'flow' of algebraic information, with the braiding $\Psi = \times$ playing the role of usual transpositions between independent symbols in ordinary mathematics, or of super transposition between independent symbols in super constructions. This menas, however, that in making braided versions of classical constructions we have to make choices of under- or over-braid crossing. Whatever construction we do, we could make a parallel one in the braided category with 'conjugate' or inverse braiding where the role of under- and over-braid crossings is reversed [17]. The idea is that in the braided approach our usual classical geometry *splits* into *two* braided versions related by a combined braid-reversal and $*$ symmetry. This is a new phenomenon not visible classically or even in super geometry, where $\Psi^2 = \mathrm{id}$. Note that these braids do not live in physical space but in the three-dimensional 'lexicographical space' in which we write our mathematical constructions diagrammatically.

The idea of $*$ mapping between two versions of q-deformed geometry rather than being (as more usual) a property of one system, is evident in the theory of $*$-structures on linear spaces developed in [8], where, for example, we viewed $* : \Omega_{\mathrm{L}} \to \Omega_{\mathrm{R}}$ between left and right versions of the q-deformed exterior algebra. We have the same phenomenon for q-deformed Poincaré quantum group function algebras. In the present context of the Poincaré enveloping algebra quantum groups it means that we have two natural coproducts, connected by $*$. This means in turn that we have two natural representations of each q-Poincaré enveloping algebra by braided differentiation on the q-space–time co-ordinates.

We can use either the differentials ∂ from [22] or 'conjugate' ones $\bar{\partial}$ defined with the inverse braiding in their braided-Leibniz rules. This makes contact with examples of 'conjugate' derivatives in [33] and elsewhere, as well as with [34] where a 'zero-dimensional' toy model for a Poincaré quantum group with two coproducts related by $*$ was considered. The new feature in our approach is that these derivatives are constructed quite systematically from the braided coaddition (from left and right) and as such we now obtain a precise and complete general understanding of how they are related by $*$ and with each other. The first main result, in Section 2, is that these two conjugate representations are isomorphic, being intertwined as

$$\underline{S}\circ\partial^i=-\bar{\partial}^i\circ\underline{S}$$

by the braided antipode $\underline{S}$ or *quantum parity operator* $\boldsymbol{x}\to-\boldsymbol{x}$ on our q-space–time coordinates. The second main result, in Section 4, is a general construction for a sesquilinear form or 'inner product' with respect to which the fundamental representation by translation and rotation is unitary. A general feature is that it is no longer exactly conjugate-symmetric but only, in case of q-Euclidean and q-Minkowski spaces, up to a power of q. We obtain in principle a braided version of the L^2 inner product on braided linear spaces, such that ∂ and $\bar{\partial}$ are mutually adjoint (or such that ∂ is self-adjoint up to $\underline{S}$). The computation of such inner products and development of the attendant 'braided analysis' are a direction for further work.

We note that some of the R-matrix formulae in the present paper can (once found) be partially verified by direct calculations using the quantum Yang–Baxter equations (QYBE) many times. This would not, however, check the various other non-R-matrix relations among quantum group generators $\boldsymbol{t}, \boldsymbol{l}^{\pm}$ etc., in the notation of [32]. For a rigorous treatment that includes all such details automatically, the abstract setting which we use in the present paper is really needed. We use it in Sections 2 and 4 where we develop elements of the abstract bosonisation theory from [15], which in turn ensures consistency with respect to all such additional relations in the R-matrix formulae presented in Sections 3 and 5.

1.1. Preliminaries

We assume that the reader is familiar with the definition of a quasitriangular Hopf algebra H in [1] with quasitriangular structure or 'universal R-matrix' $\mathcal{R}=\mathcal{R}^{(1)}\otimes\mathcal{R}^{(2)}$ in $H\otimes H$ (summation of terms implicit), and the dual notion of a dual-quasitriangular Hopf algebra A with dual-quasitriangular structure or 'universal R-matrix funtional' $\mathcal{R}: A\otimes A\to\mathbb{C}$ in [35,36]. The latter is characterised by the axioms

$$\begin{aligned}&\mathcal{R}(a\otimes bc)=\mathcal{R}(a_{(1)}\otimes c)\mathcal{R}(a_{(2)}\otimes b),\\&\mathcal{R}(ab\otimes c)=\mathcal{R}(a\otimes c_{(1)})\mathcal{R}(b\otimes c_{(2)}),\\&b_{(1)}a_{(1)}\mathcal{R}(a_{(2)}\otimes b_{(2)})=\mathcal{R}(a_{(1)}\otimes b_{(1)})a_{(2)}b_{(2)}.\end{aligned}\tag{1}$$

The coproducts are denoted $\Delta a=a_{(1)}\otimes a_{(2)}$, etc. (summation implicit).

It is well-known that the category of representations (modules) of a quasitriangular Hopf algebra forms a braided category with braiding $\Psi=\times$ given by acting via $\mathcal{R}$ and then making the usual transposition of the underlying vector spaces [1,13,37]. By braided category we mean a collection of objects with a tensor product which is associative and commutative up to isomorphism. We suppress the associativity isomorphism (which is trivial in our examples) and write the commutativity isomorphism as the braiding Ψ. There are various coherence axioms between these structures to the effect that the rules for working in a braided category are the obvious ones suggested by the braid-crossing notation. The category of corepresentations (comodules) of a dual-quasitriangular Hopf algebra likewise forms a braided category, with Ψ given by coacting on each comodule, evaluating the relevant outputs against $\mathcal{R}$ and making the usual transposition of the underlying vector spaces [35,36]. Introductions are in [18].

The notion of duality of Hopf algebras is best handled for our purposes as a duality pairing [13] between two Hopf algebras rather than regarding one as a subspace of linear functionals on the other. The product of one maps as usual to the coproduct of the other,

$$\begin{aligned}\langle h, ab\rangle &= \langle h_{(1)}, a\rangle\langle h_{(2)}, b\rangle,\\ \langle hg, a\rangle &= \langle h, a_{(1)}\rangle\langle g, a_{(2)}\rangle,\\ \langle h, Sa\rangle &= \langle Sh, a\rangle\end{aligned} \tag{2}$$

for all $a, b \in A,\ h,\ g \in H$. The axioms of a Hopf $*$-algebra have been studied extensively by Woronowicz [10] and are that our Hopf algebra is a $*$-algebra and

$$(*\otimes *)\circ\Delta = \Delta\circ *,\qquad \overline{\epsilon(\ )} = \epsilon\circ *,\qquad *\circ S = S^{-1}\circ *. \tag{3}$$

When we have a quasitriangular structure, it is natural to require $\mathcal{R}^{*\otimes *} = \mathcal{R}_{21}$ or $\mathcal{R}^{*\otimes *} = \mathcal{R}^{-1}$ as explained in [25,38] among other places. The first type is called *real quasitriangular*.

We also assume that the reader is familiar with the basic notion of a braided group B or braided-Hopf algebra [36,39]. Introductions are in [17–19]. A braided group B is like a quantum group but the coproduct $\underline{\Delta} : B \to B\underline{\otimes}B$ maps to the *braided tensor product* where the two factors in the tensor product do not commute. Instead they enjoy mutual *braid statistics*. In mathematical terms we have a Hopf algebra in a braided category, where the braiding Ψ between any two objects determines their mutual braided tensor product algebra. There are also more usual axioms for a braided antipode $\underline{S} : B \to B$ and counit $\underline{\epsilon} : B \to \mathbb{C}$. The diagrammatic way of working with braided groups consists of writing all maps as arrows generally pointing downwards. We write tensor products of objects by horizontal juxtaposition, $\Psi = \times$ as a braid crossing and Ψ^{-1} as the reversed braid crossing. We write other morphisms as nodes with appropriate valency. So the product morphism of a braided group is $\cdot = \curlyvee$ and the coproduct morphism is $\underline{\Delta} = \curlywedge$. Functoriality of the braiding is expressed as being allowed to pull nodes through braid crossings as if they are beads on the string or tangle. The appropriate coherence theorem ensures that these rules are consistent and that 'topologically equivalent' diagrams correspond to the same algebraic operations. This kind of 'braided algebra' appeared in [15] and is a characteristic feature of the theory of braided groups [17,18].

In practice, one can also work with braided groups without being too categorical: one can simply specify every time the required cross relations or braid statistics in the braided tensor product $B\underline{\otimes}B$. We write $b \equiv b \otimes 1$ for elements in the first factor and $b' \equiv 1 \otimes b$ for elements of the second, and give the relations between b, b' [3]. The abstract theory of braided categories is nevertheless needed to ensure that all these different braided tensor products are mutually consistent or 'coherent'. Otherwise the idea would be too general and probably intractable.

The appropriate axioms for a $*$-braided group were introduced in [25] for a large class of braided groups and confirmed in [8] for linear spaces. We require that our braided group is a $*$-algebra and

$$(*\otimes *)\circ \underline{\Delta} = \tau \circ \underline{\Delta} \circ *, \qquad \overline{\underline{\epsilon}(\ )} = \underline{\epsilon} \circ *, \qquad * \circ \underline{S} = \underline{S} \circ *, \tag{4}$$

where τ is the usual transposition. The duality pairing $\mathrm{ev} = \cup$ for braided groups B, C is [17]

$$\begin{aligned} \mathrm{ev}\,(ab, c) &= \mathrm{ev}\,(a, c_{\underline{(2)}})\,\mathrm{ev}(b, c_{\underline{(1)}}),\\ \mathrm{ev}\,(b, cd) &= \mathrm{ev}(b_{\underline{(2)}}, c)\,\mathrm{ev}\,(b_{\underline{(1)}}, d),\\ \mathrm{ev}\,(\underline{S}b, c) &= \mathrm{ev}\,(b, \underline{S}c) \end{aligned} \tag{5}$$

for all $a, b \in B, c, d \in C$. Here braided coproducts are denoted $\underline{\Delta}b = b_{\underline{(1)}} \otimes b_{\underline{(2)}}$, etc. (summation implicit). Equivalently (if the braided antipode is invertible) we can use $\langle\ ,\ \rangle \equiv \mathrm{ev}\,(\underline{S}^{-1}(\ ), (\ ))$ obeying

$$\begin{aligned} \langle ab, c\rangle &= \langle a, \Psi(b \otimes c_{\underline{(1)}}), c_{\underline{(2)}}\rangle,\\ \langle b, cd\rangle &= \langle b_{\underline{(1)}}, \Psi(b_{\underline{(2)}} \otimes c), d\rangle,\\ \langle \underline{S}b, c\rangle &= \langle b, \underline{S}c\rangle, \end{aligned} \tag{6}$$

where we apply Ψ and evaluate its left-hand output with $\langle a,\ \rangle$ and its right-hand output with $\langle\ , c_{\underline{(2)}}\rangle$, etc. This follows from the braided-antihomomorphism property [17]

$$\underline{S} \circ \cdot = \cdot \circ \Psi \circ (\underline{S} \otimes \underline{S}), \qquad \underline{\Delta} \circ \underline{S} = (\underline{S} \otimes \underline{S}) \circ \Psi \circ \underline{\Delta} \tag{7}$$

for $\underline{S}$ and similarly (with Ψ^{-1}) for $\underline{S}^{-1}$. It is to avoid the extra braiding that we generally prefer (5). In the case of strict duality where $B = C^\star$ the map $\mathrm{ev} : C^\star \otimes C \to \mathbb{C}$ comes also with a coevaluation $\mathrm{coev} = \cap$, making C a rigid object in the braided category. When we have $*$-structures, their natural axioms under duality for quantum groups and braided groups are

$$\overline{\langle h, a\rangle} = \langle (Sh)^*, a\rangle, \qquad \overline{\mathrm{ev}(b, c)} = \mathrm{ev}\,(b^*, c^\star) \tag{8}$$

according to [10] and [8] respectively. The $*$-operation $c^\star$ compatible in this way is not necessarily 'unitary' in a natural sense but is often expressible in terms of second $*$-structure c^* which is. This was already noted for braided linear spaces in [8] and will play a role in Sections 4 and 5.

Among the main theorems about braided groups is that if B is any braided group living in the braided category of H-modules (H quasitriangular), then there is an ordinary quantum group $B{>\!\!\!\triangleleft} H$ called its *bosonisation* and characterised abstractly such that the ordinary representations of $B \rtimes\!\!\!\cdot H$ are in 1–1 correspondence with the braided (H-covariant) representations of B [15]. Explicitly, $B \rtimes\!\!\!\cdot H$ is generated as an algebra by H, B and has cross relations, coproduct and antipode

$$hb = (h_{(1)} \triangleright b)h_{(2)}, \quad \Delta b = b_{\underline{(1)}}\mathcal{R}^{(2)} \otimes \mathcal{R}^{(1)} \triangleright b_{\underline{(2)}}, \quad Sb = (u\mathcal{R}^{(1)} \triangleright \underline{S}b)S\mathcal{R}^{(2)} \tag{9}$$

for all $h \in H, b \in B$. Here $u = (S\mathcal{R}^{(2)})\mathcal{R}^{(1)}$ is an element of H. The algebra is a standard semidirect product by the canonical action $\triangleright$ of H on B, while the coalgebra is a semidirect coproduct by the coaction induced by the universal R-matrix as explained in [20]. Likewise if B lives in the category of A-comodules (A dual quasitriangular) then there is an ordinary quantum group $A \cdot\!\!\!\ltimes B$ called its *cobosonisation*, such that the ordinary corepresentations of $A \cdot\!\!\!\ltimes B$ are in 1–1 correspondence with the braided (A-covariant) corepresentations of B [25]. Explicitly, $A \cdot\!\!\!\ltimes B$ is generated as an algebra by B, A and has cross relations, coproduct and antipode

$$ba = a_{(1)}b^{(\bar{1})}\mathcal{R}(b^{(\bar{2})} \otimes a_{(2)}), \quad \Delta b = b_{\underline{(1)}}{}^{(\bar{1})} \otimes b_{\underline{(1)}}{}^{(\bar{2})}b_{\underline{(2)}}, \quad Sb = (\underline{S}b^{(\bar{1})})Sb^{(\bar{2})} \tag{10}$$

for all $a \in A, b \in B$. This time the coalgebra is a semidirect coproduct by the coaction of A on B, which we denote $b^{(\bar{1})} \otimes b^{(\bar{2})}$ for the resulting element of $B \otimes A$ (summation implicit).

Finally, we recall a theory of twisting of quasiquantum groups due to Drinfeld [16]. A special case of it implies at once that if H is a quasitrinagular Hopf algebra and $\chi \in H \otimes H$ a *quantum* 2-*cocycle* in the sense

$$\chi_{12}(\Delta \otimes \mathrm{id})\chi = \chi_{23}(\mathrm{id} \otimes \Delta)\chi, \quad (\epsilon \otimes \mathrm{id})\chi = 1. \tag{11}$$

then H_χ defined by [16]

$$\Delta_\chi = \chi(\Delta)\chi^{-1}, \quad \epsilon_\chi = \epsilon, \quad \mathcal{R}_\chi = \chi_{21}\mathcal{R}\chi^{-1}, \quad S_\chi = U(S)U^{-1}, \tag{12}$$

where $U = \chi^{(1)}S\chi^{(2)}$, is again a quasitriangular Hopf algebra. The cohomological terminology in this context is justified in [40]. This specialisation of Drinfeld's ideas was studied in [41], and using the formula for ΔU given there, it is easy to see that if H is a Hopf $*$-algebra and $(S \otimes S)(\chi^{*\otimes *}) = \chi_{21}$ then

$$*_\chi = (S^{-1}U)((\ )^*)S^{-1}U^{-1} \tag{13}$$

makes H_χ into a Hopf $*$-algebra as well. We will see this in more detail in the course of a proof in Section 2. The purpose of [41] was to consider how the corresponding braided groups constructed from H, H_χ by transmutation [39] are related. We will consider the adjoint of this question, namely a twisting theory of braided groups such that their bosonisations are related by quantum group twisting as above. We also consider how $*$ interacts with the bosonisation construction.

The abstract sections (Sections 2 and 4) in the paper work over a general field or (with suitable care) a commutative ring for purely algebraic results, and over $\mathbb{C}$ when we discuss $*$. The specific examples in Section 3 based on quantum enveloping algebras work over formal powerseries $\mathbb{C}[[t]]$ in a deformation parameter whenever we require directly the quasitriangular structure $\mathcal{R}$, in the standard way [1]. (The Hopf algebras themselves, their $*$-structures and their representations do not require this, however, and all work over $\mathbb{C}$ when we use suitable generators, again in the standard way.) All antipodes and braided antipodes are required for convenience to be invertible, which is in any case automatic for quasitriangular and dual-quasitriangular Hopf algebras.

2. $*$-Structure and twisting of bosonisations

In this section we refine some of the abstract results on the bosonisation construction [15]. This construction can then be used to define inhomogeneous quantum groups as shown in [9], such as q-Poincaré quantum groups where the momentum sector is a braided covector space (a linear braided group) living in the braided category of q-Lorentz corepresentations (i.e., it is q-Lorentz covariant). Bosonisation consists of adjoining the q-Lorentz sector as a particular semidirect product. Another example of bosonisation is for a super-Hopf algebra in the category of super-vector spaces generated by a certain quantum group $\mathbb{Z}'_2$. This is like the Jordan–Wigner transform which consists in adjoining a degree operator and thereby rendering a fermionic or super system into a bosonic one. These two settings, Lorentz covariance and super symmetry, are mathematically unified as cases of one construction. As well as the Poincaré quantum group, other interesting applications are to super symmetry [42,43] and to the theory of differential calculus on quantum groups [44,45]. For the moment we proceed quite generally.

Firstly, it should be perfectly clear that the bosonisation and cobosonisation constructions (9) and (10) are conceptually dual to one another. The bosonisation in [15] was constructed diagrammatically by a braided group semidirect product construction to give a certain braided Hopf algebra. The Hopf algebra $B \rtimes\!\!\!\cdot H$ contains H and is arranged so that transmutation from this inclusion [46] reconstructs this braided-Hopf algebra. This step is also diagrammatic. By turning all diagrams up-side-down one gets the dual construction which gives (10). We make a braided semidirect coproduct and use the dual transmutation theory in [35,36]. On the other hand, since not all readers will be comfortable with the diagrammatic theory, we check the duality now quite explicitly. Once we have the relevant formulae in detail, we concentrate with just the bosonisation (9), leaving the corresponding dual results for the cobosonisation as an easy exercise.

To this end, suppose that H, A are dually paired quantum groups as in (2) with corresponding quasitriangular and dual-quasitriangular structure. Let B be a braided group in the category of H-modules and C a braided group in the category of A-comodules, which when viewed in the category of H-modules is dually paired to B as in (5).

Lemma 2.1. *With A, H and B, C dually paired as stated, the two ordinary Hopf algebras $B \rtimes\!\!\!\cdot H$ and $A \cdot\!\!\!\ltimes C$ are dually paired. Between the various subalgebras the pairing is*

$$\langle b, a\rangle = \epsilon(b)\epsilon(a), \qquad \langle h, a\rangle = \text{usual},$$
$$\langle b, c\rangle = \mathrm{ev}\,(\underline{S}^{-1}b, c), \qquad \langle h, c\rangle = \epsilon(h)\epsilon(c)$$

for all $a \in A, h \in H, b \in B, c \in C$.

Proof. The pairing between subalgebras extends uniquely by conditions (2) to the pairing

$$\langle bh, ac\rangle = \langle \mathcal{R}^{-(2)} h_{(1)}, a\rangle \mathrm{ev}\,(\underline{S}^{-1}b, \mathcal{R}^{-(1)} h_{(2)} \triangleright c) \tag{14}$$

between general elements. We check that this is indeed a pairing. We write $\langle b, c\rangle = \mathrm{ev}\,(\underline{S}^{-1}b, c)$ and need only properties (6) mentioned in Section 1.1, so in fact this lemma works if we suppose $\langle\ ,\ \rangle$ directly without necessarily supposing $\underline{S}^{-1}$. We denote further distinct copies of $\mathcal{R}$ by $\mathcal{R}'$, etc., and consider general $b, d \in B$, $h, g \in H$, $a \in A$ and $c \in C$. Then

$$\begin{aligned}
&\langle bh \otimes dg, \Delta(ac)\rangle \\
&\quad = \langle bh, a_{(1)} c_{\underline{(1)}}{}^{(\bar 1)}\rangle \langle dg, a_{(2)} c_{\underline{(1)}}{}^{(\bar 2)} c_{\underline{(2)}}\rangle \\
&\quad = \langle \mathcal{R}^{-(2)} h_{(1)}, a_{(1)}\rangle \langle \mathcal{R}'^{-(2)} g_{(1)}, a_{(2)} c_{\underline{(1)}}{}^{(\bar 2)}\rangle \langle b, \mathcal{R}^{-(1)} h_{(2)} \triangleright c_{\underline{(1)}}{}^{(\bar 1)}\rangle \\
&\qquad \times \langle d, \mathcal{R}'^{-(1)} g_{(2)} \triangleright c_{\underline{(2)}}\rangle \\
&\quad = \langle \mathcal{R}^{-(2)} h_{(3)} \mathcal{R}'^{-(2)}{}_{(1)} g_{(1)}, a\rangle \langle b, \mathcal{R}^{-(1)} h_{(4)} \mathcal{R}'^{-(2)}{}_{(2)} g_{(2)} \triangleright c_{\underline{(1)}}\rangle \\
&\qquad \times \langle h_{(1)} \triangleright d, h_{(2)} \mathcal{R}'^{-(1)} g_{(3)} \triangleright c_{\underline{(2)}}\rangle \\
&\quad = \langle \mathcal{R}'^{-(2)} h_{(2)} g_{(1)}, a\rangle \langle b, \mathcal{R}^{-(2)} \mathcal{R}'^{-(1)}{}_{(1)} h_{(3)} g_{(2)} \triangleright c_{\underline{(1)}}\rangle \\
&\qquad \times \langle h_{(1)} \triangleright d, \mathcal{R}^{-(1)} \mathcal{R}'^{-(1)}{}_{(2)} h_{(4)} g_{(3)} \triangleright c_{\underline{(2)}}\rangle \\
&\quad = \langle \mathcal{R}'^{-(2)} h_{(2)} g_{(1)}, a\rangle \langle b, \mathcal{R}^{-(2)} \triangleright (\mathcal{R}'^{-(1)} h_{(3)} g_{(2)} \triangleright c)_{\underline{(1)}}\rangle \\
&\qquad \times \langle h_{(1)} \triangleright d, \mathcal{R}^{-(1)} \triangleright (\mathcal{R}'^{-(1)} h_{(3)} g_{(2)} \triangleright c)_{\underline{(2)}}\rangle \\
&\quad = \langle \mathcal{R}^{-(2)} h_{(2)} g_{(1)}, a\rangle \langle b(h_{(1)} \triangleright d), \mathcal{R}^{-(1)} h_{(3)} g_{(2)} \triangleright c\rangle \\
&\quad = \langle b(h_{(1)} \triangleright d) h_{(2)} g, ac\rangle = \langle bhdg, ac\rangle,
\end{aligned}$$

where the first equality is the coproduct from (10), the second evaluates (14), the third uses the Hopf algebra pairing (2) and also writes evaluation of the coaction of $c_{\underline{(1)}}$ as a further action on it. At the same time we insert an extra $h_{(1)}$ acting on d, and $h_{(2)}$ on the other input of $\langle\ ,\ \rangle$, knowing that this is trivial since ev and $\langle\ ,\ \rangle$ between the braided groups are H-invariant. For the fourth equality we use $(h_{(2)} \otimes h_{(3)})\mathcal{R}' = \mathcal{R}'(h_{(3)} \otimes h_{(2)})$ and then the cocycle properties of $\mathcal{R}$ from the axioms of a quasitriangular Hopf algebra. The fifth equality is the covariance of the braided coproduct of C. Finally, we use the braided group duality in the form (6) and recognise the result as the product from (9). This is half of the lemma. For the other half, we verify on $b \in B$, $h \in H$, $a, d \in A$ and $c, e \in C$,

$$
\begin{aligned}
&\langle \Delta(bh), ac\otimes de\rangle\\
&= \langle b_{\underline{(1)}}\mathcal{R}^{(2)}h_{(1)}, ac\rangle\langle(\mathcal{R}^{(1)}\triangleright b_{\underline{(2)}})h_{(2)}, de\rangle\\
&= \langle \mathcal{R}'^{-(2)}\mathcal{R}^{-(2)}{}_{(1)}h_{(1)}, a\rangle\langle\mathcal{R}''^{-(2)}h_{(3)}, d\rangle\langle b_{\underline{(1)}}, \mathcal{R}'^{-(1)}\mathcal{R}^{-(2)}{}_{(2)}h_{(2)}\triangleright c\rangle\\
&\quad\times\langle b_{\underline{(2)}}, \mathcal{R}^{-(1)}\mathcal{R}''^{-(1)}h_{(4)}\triangleright e\rangle\\
&= \langle \mathcal{R}^{-(2)}h_{(1)}, a\rangle\langle h_{(4)}\mathcal{R}''^{-(2)}, d\rangle\langle b_{\underline{(1)}}, \mathcal{R}'^{-(2)}\mathcal{R}^{-(1)}{}_{(1)}h_{(2)}\triangleright c\rangle\\
&\quad\times\langle b_{\underline{(2)}}, \mathcal{R}'^{-(1)}\mathcal{R}^{-(1)}{}_{(2)}h_{(3)}\mathcal{R}''^{-(1)}\triangleright e\rangle\\
&= \langle \mathcal{R}^{-(2)}h_{(1)}, a\rangle\langle h_{(4)}\mathcal{R}''^{-(2)}, d\rangle\langle b, (\mathcal{R}^{-(1)}{}_{(1)}h_{(2)}\triangleright c)(\mathcal{R}^{-(1)}{}_{(2)}h_{(3)}\mathcal{R}''^{-(1)}\triangleright e)\rangle\\
&= \langle \mathcal{R}^{-(2)}{}_{(1)}h_{(1)}, a\rangle\langle\mathcal{R}^{-(2)}{}_{(2)}h_{(2)}\mathcal{R}^{(2)}, d\rangle\\
&\quad\times\langle b, (\mathcal{R}^{-(1)}{}_{(1)}h_{(3)}\mathcal{R}^{(1)}\triangleright c)(\mathcal{R}^{-(1)}{}_{(2)}h_{(4)}\triangleright e)\rangle\\
&= \langle \mathcal{R}^{-(2)}h_{(1)}, ad_{(1)}\rangle\langle\mathcal{R}^{(2)}, d_{(2)}\rangle\langle b, \mathcal{R}^{-(1)}h_{(2)}\triangleright((\mathcal{R}^{(1)}\triangleright c)e)\rangle\\
&= \langle bh, ad_{(1)}c^{(\bar{1})}e\rangle\mathcal{R}(c^{(\bar{2})}\otimes d_{(2)}) = \langle bh, acde\rangle,
\end{aligned}
$$

where the first equality is the coproduct from (9), the second evaluates (14) and also moves $\mathcal{R}^{(1)}$ acting on $b_{\underline{(2)}}$ to $(S\mathcal{R}^{(1)})$ (which becomes $\mathcal{R}^{-(1)}$) acting on the other input of the braided group pairing (by its H-invariance). The third equality uses the cocycle property of $\mathcal{R}^{-1}$ coming from the axioms of a quasitriangular Hopf algebra, and also moves $h_{(3)}\otimes h_{(4)}$ to the left of $\mathcal{R}''$ as $h_{(4)}\otimes h_{(3)}$ using these axioms. The fourth equality uses the braided group duality (6). The fifth equality uses the cocycle and other quasitriangularity axioms for $\mathcal{R}$ again. The sixth uses the usual duality (2) and covariance of the braided group product. We then use (10) to recognise the result. □

Lemma 2.1 reworks [25] where the duality was given explicitly when $C = B^\star$ rather than $B = C^\star$ as it is here: Both statements are true. The astute reader will note also that if these constructions are dual and both A and H are sub-Hopf algebras then both A, H are also projected onto. Thus both $A \bowtie C$ and $B \bowtie H$ are Hopf algebras with projection in the sense of Radford [47]. This observation is due to the author in [20].

Lemma 2.1 is useful when making explicit dualisations between the comodule and module setting. For example, the coproduct in (10) can obviously be considered as a coaction of $A \bowtie C$ on C from the right. It is how the quantum Poincaré function algebras in [9] coact on the space–time co-ordinate algebra. Dualising means that in the setting above, $B \bowtie H$ acts on the space–time co-ordinate algebra.

Corollary 2.2. *In the setting of Lemma 2.1, C is a left $B \bowtie H$-module algebra by*

$$h\triangleright c = c^{(\bar{1})}\langle h, c^{(\bar{2})}\rangle,\qquad b\triangleright c = \mathcal{R}^{-(2)}\triangleright c_{\underline{(1)}}\mathrm{ev}\,(\underline{S}^{-1}b, \mathcal{R}^{-(1)}\triangleright c_{\underline{(2)}}) \tag{15}$$

for $h\in H$, $b\in B$ and $c\in C$. We call this the fundamental representation of the bosonised Hopf algebra.

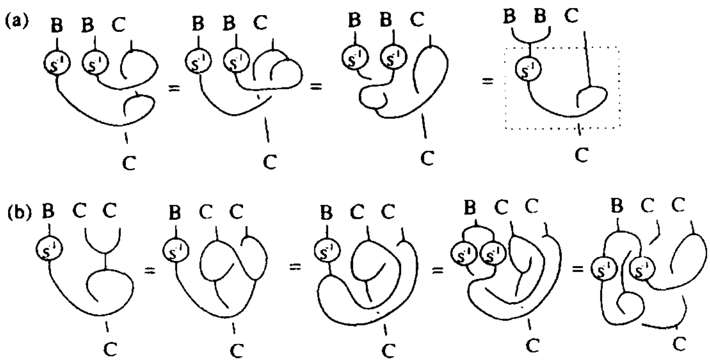


Fig. 1. Proof that the action of B on C in Corollary 2.2 (in box) is (a) an action and (b) braided covariant.

Proof. Conceptually, the action of H on C is just the action corresponding to the coaction of A assumed when we said that B was A-covariant to begin with. The action of B in abstract terms is

$$b \triangleright c = (\mathrm{ev}(\underline{S}^{-1}b, (\ )) \otimes \mathrm{id}) \circ \Psi^{-1} \circ \underline{\Delta}c, \tag{16}$$

which has a diagrammatic picture when we write Ψ as a braid crossing and $\mathrm{ev} = \cup$. This is shown in the box in Fig. 1, where we check that it indeed makes C a braided-module algebra under its dual B. Part (a) checks that it is an action, using coassociativity of the braided coproduct of C, dualising it to a product of B and the anti-algebra homomorphism property of $\underline{S}^{-1}$ proven in [17]. Part (b) checks that we have a braided B-module algebra in the sense of [15,17], using the braided bialgebra axiom for C, dualising one of its products to a coproduct of B, and the anti-coalgebra homomorphism property of $\underline{S}^{-1}$. The role of $\mathrm{ev} \circ \underline{S}^{-1}$ can be played directly by a braided group duality pairing of the type (6) if we prefer. The 1–1 correspondence of representations in bosonisation theory then means that C becomes an ordinary module algebra under the bosonisation of B when we adjoin H. One gets the same answer with more work by explicitly evaluating against the coproduct of $A \bowtie C$ viewed as a coaction, via the duality pairing from Lemma 2.1. □

If B is a braided group then its *naive opposite coproduct* $\Psi^{-1} \circ \underline{\Delta}$ makes the algebra of B into a braided group B^{cop} living not in our original braided category but rather in the 'conjugate' braided category with inverse braiding [17, Lemma 4.6]. $\underline{S}^{-1}$ becomes its braided antipode. In concrete terms it means that the braided group B^{cop} is no longer properly covariant under H (with the correct induced braiding) but under this quantum group equipped with $\mathcal{R}_{21}^{-1}$ instead for its universal R-matrix. Let us denote the latter by $\bar{H}$. As a Hopf algebra it coincides with H, but has 'conjugate' $\mathcal{R}$. So we can apply our bosonisation theorem (10) and obtain at once a new Hopf algebra $B^{\mathrm{cop}} \rtimes\!\!\!\cdot\, \bar{H}$ with coproduct and antipode $\bar{\Delta}$, $\bar{S}$ say. As an algebra it coincides with $B \rtimes\!\!\!\cdot\, H$ so $\bar{\Delta}$ is a second 'conjugate' Hopf algebra structure in this same algebra.

Proposition 2.3. *Let $B{>\!\!\!\triangleleft} H$ be the bosonisation of a braided group B. The second 'conjugate' coproduct and antipode on the same algebra is*

$$\bar{\Delta} b=\mathcal{R}^{-(1)} b_{\underline{(2)}} \otimes \mathcal{R}^{-(2)} \triangleright b_{\underline{(1)}}, \qquad \bar{S} b=(\mathcal{R}^{(2)} v^{-1} \triangleright \underline{S}^{-1} b) \mathcal{R}^{(1)} \tag{17}$$

where $v=\mathcal{R}^{(1)} S \mathcal{R}^{(2)}$, and is twisting equivalent to $(B{>\!\!\!\triangleleft} H)^{\mathrm{cop}}$ by quantum 2-cocycle $\mathcal{R}^{-1}$.

Proof. We compute from (9) for our new braided group B^{cop} with opposite coproduct $\Psi^{-1} \circ \Delta b=\mathcal{R}^{-(1)} \triangleright b_{\underline{(2)}} \otimes \mathcal{R}^{-(2)} \triangleright b_{\underline{(1)}}$ and quantum group $\bar{H}$ with quasitriangular structure $\mathcal{R}_{21}^{-1}$. Then (9) gives bosonised coproduct

$$\begin{aligned}
\bar{\Delta} b &=(\mathcal{R}^{-(1)} \triangleright b_{\underline{(2)}}) \mathcal{R}^{\prime-(1)} \otimes(\mathcal{R}^{\prime-(2)} \mathcal{R}^{-(2)}) \triangleright b_{\underline{(1)}} \\
&=(\mathcal{R}^{-(1)}{}_{(1)} \triangleright b_{\underline{(2)}}) \mathcal{R}^{-(1)}{}_{(2)} \otimes \mathcal{R}^{-(2)} \triangleright b_{\underline{(1)}}
\end{aligned}$$

using one of the axioms of a quasitriangular structure. We recognise the result as stated in the proposition when we use the product (9) in $B{>\!\!\!\triangleleft} H$. The antipode is likewise computed from (9) for our new braided group B^{cop}, which has braided antipode $\underline{S}^{-1}$ [17]. We can also compute $\bar{\Delta}$ further as

$$\begin{aligned}
\bar{\Delta} b &=\mathcal{R}^{-(1)} \mathcal{R}^{\prime-(1)} \mathcal{R}^{(1)} b_{\underline{(2)}} \otimes(\mathcal{R}^{-(2)} \triangleright b_{\underline{(1)}}) \mathcal{R}^{\prime-(2)} \mathcal{R}^{(2)} \\
&=\mathcal{R}^{-(1)} \mathcal{R}^{(1)} b_{\underline{(2)}} \otimes(\mathcal{R}^{-(2)}{}_{(1)} \triangleright b_{\underline{(1)}}) \mathcal{R}^{-(2)}{}_{(2)} \mathcal{R}^{(2)} \\
&=\mathcal{R}^{-(1)} \mathcal{R}^{(1)} b_{\underline{(2)}} \otimes \mathcal{R}^{-(2)} b_{\underline{(1)}} \mathcal{R}^{(2)} \\
&=\mathcal{R}^{-(1)}(\mathcal{R}^{(1)}{}_{(1)} \triangleright b_{\underline{(2)}}) \mathcal{R}^{(1)}{}_{(2)} \otimes \mathcal{R}^{-(2)} b_{\underline{(1)}} \mathcal{R}^{(2)} \\
&=\mathcal{R}^{-(1)}(\mathcal{R}^{(1)} \triangleright b_{\underline{(2)}}) \mathcal{R}^{\prime(1)} \otimes \mathcal{R}^{-(2)} b_{\underline{(1)}} \mathcal{R}^{(2)} \mathcal{R}^{\prime(2)},
\end{aligned}$$

which we recognise as $\mathcal{R}^{-1}(\tau \circ \Delta b) \mathcal{R}$ in view of the coproduct from (9). The first equality inserts $\mathcal{R}^{-1} \mathcal{R}$ into our previous result for $\bar{\Delta} b$. The second uses quasitriangularity of H, the third uses the relations in (9) on the right-hand factor. We then use relations (9) on the left-hand factor for the fourth equality and quasitriangularity again for the fifth. Note that $\bar{\Delta} h=\Delta h$ since H is a sub-Hopf algebra, which also equals $\mathcal{R}^{-1}(\tau \circ \Delta h) \mathcal{R}$ by quasitriangularity. The quasitriangularity axioms imply that $\mathcal{R}^{-1}$ is a 2-cocycle for H^{cop} in the sense of (11), but since H is a sub-Hopf algebra of $B{>\!\!\!\triangleleft} H$, we can also view it as a 2-cocycle for $(B{>\!\!\!\triangleleft} H)^{\mathrm{cop}}$. We see that the second 'conjugate' Hopf algebra structure on $B{>\!\!\!\triangleleft} H$ is the twisting of $(B{>\!\!\!\triangleleft} H)^{\mathrm{cop}}$ by this cocycle. The antipodes are also twisted one into the other, since they are determined by the coproducts. □

Both the twisting and the 'conjugate' point of view on this second coproduct are useful.

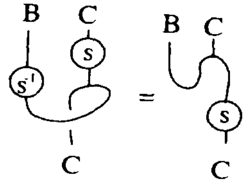
Corollary 2.4. *In the dual pairing setting of Lemma 2.1, C is a module-algebra under $B{>\!\!\!\triangleleft} H$ with its second 'conjugate' Hopf algebra structure, via*

$$h \bar{\triangleright} c=c^{(\bar{1})}\langle h, c^{(\bar{2})}\rangle, \qquad b \bar{\triangleright} c=\operatorname{ev}(b, c_{\underline{(1)}}) c_{\underline{(2)}} \tag{18}$$

for $h \in H, b \in B$ *and* $c \in C$. *Moreover, this* conjugate fundamental representation *is isomorphic to the representation in Corollary* 2.2 *via the braided antipode of C as intertwiner,*

$$\underline{S}(x\bar{\triangleright}c) = x \triangleright \underline{S}c, \quad \forall x \in B{>\!\!\triangleleft} H.$$

Proof. We deduce this without computation via 'braid' crossing reversal symmetry', by applying Corollary 2.2 to $B^{\rm cop}$ in the category with inverse braiding. The role of C is now played by $C^{\rm op}$ with opposite product $\cdot \circ \Psi^{-1}$. The braided antipode of C is [17] an isomorphism $\underline{S} : C^{\rm op} \to C^{\rm cop}$ of braided groups and we use this now to refer our action to $C^{\rm cop}$. Then (16) uses the opposite coproduct to the coproduct $\Psi^{-1} \circ \underline{\Delta}$ which (in our category with reversed braiding) is $\Psi \circ \Psi^{-1} \circ \underline{\Delta} = \underline{\Delta}$. $\underline{S}^{-1}$ in (16) becomes the inverse $\underline{S}$ of the antipode of B and is absorbed in the above isomorphism. In fact, the resulting action is exactly the left-translation used in defining braided differentiation in [22], and we know directly from there that it makes C a braided $B^{\rm cop}$-module algebra in the category with reversed braiding (it has the inverse braiding in its Leibniz rule). It then bosonises to an action of $B{>\!\!\triangleleft} H$ by adjoining the action of H. Moreover, we can apply a further $\underline{S}$ to $C^{\rm cop}$ and then our action of B becomes on its image the representation in Corollary 2.2. Thus



using (7). The action of H in the two cases is the same, namely the one by which C lives in the category of H-modules, and $\underline{S}$ already intertwines this part of the action because all braided group maps are morphisms in the category. □

Next we suppose that H is a Hopf $*$-algebra and B is a $*$-braided group, and that H acts on B 'unitarily' in the standard sense

$$(h \triangleright b)^* = (Sh)^* \triangleright b^*. \tag{19}$$

Then the usual theory of Hopf algebra semidirect products ensures that $B{>\!\!\triangleleft} H$ is a $*$-algebra. See [25], where this question was considered specifically for bosonisations. So $B{>\!\!\triangleleft} H$ is certainly a $*$-algebra.

Lemma 2.5. *If H is a real-quasitriangular Hopf $*$-algebra and acts unitarily in the sense* (19) *on a $*$-braided group B in its category of representations, then $*$ intertwines the original coproduct Δ of $B{>\!\!\triangleleft} H$ and $\bar{\Delta}$, i.e.* $(* \otimes *) \circ \Delta \circ * = \bar{\Delta}$. *Likewise,* $* \circ S \circ * = \bar{S}^{-1}$.

Proof. We compute

$$\begin{aligned}(* \otimes *) \circ \Delta b &= \mathcal{R}^{(2)*} b_{\underline{(1)}}{}^* \otimes (\mathcal{R}^{(1)} \triangleright b_{\underline{(2)}})^* \\ &= \mathcal{R}^{-(1)} b_{\underline{(1)}}{}^* \otimes \mathcal{R}^{-(2)} \triangleright (b_{\underline{(2)}}{}^*) = \bar{\Delta}(b^*),\end{aligned}$$

where the second equality is our reality and unitarity assumption and the third is the $*$-axiom (3) for braided groups. The mapping over of the antipodes under $*$ as stated is then uniquely determined by the mapping over of the coproducts. □

Note that the content here is not that $(*\otimes *)\circ \Delta \circ *$ defines a Hopf algebra (it is just the Hopf algebra with opposite product, mapped over by $*$) but that it coincides with $\bar{\Delta}$ constructed by the 'conjugate' bosonisation. This manifests the deep connection between $*$ and braiding in the sense of a combined conjugation-braid reversal symmetry. Lemma 2.5 also tells us what kind of properties for $*$ to expect for quantum groups $B{>\!\!\!\triangleleft} H$ obtained by bosonisation. We see that the coproducts Δ and $\bar{\Delta}$ coincide on the quantum group part H, where they are both its usual coproduct. But on the braided group part B they are more like opposite (transposed) coproducts and indeed become that when $\mathcal{R}=1$. This is how our hybrid quantum group interpolates between axioms (4) for a braided group (with a transposition τ) and (3) for a usual quantum group (without τ). Putting this together with the twisting characterisation of $\bar{\Delta}$ above, we are motivated to define the following.

Definition 2.6. A quasi-$*$ Hopf algebra is a Hopf algebra which is a $*$-algebra such that

$$(*\otimes *)\circ \Delta\circ * = \mathcal{R}^{-1}(\tau\circ\Delta)\mathcal{R},\qquad \overline{\epsilon(\ )}=\epsilon\circ *,$$
$$(\mathrm{id}\otimes\Delta)\mathcal{R}=\mathcal{R}_{13}\mathcal{R}_{12},\quad (\Delta\otimes\mathrm{id})\mathcal{R}=\mathcal{R}_{13}\mathcal{R}_{23},\quad \mathcal{R}^{*\otimes *}=\mathcal{R}_{21}$$

for an invertible element $\mathcal{R}$ of $H\otimes H$.

We will study such objects further in Section 4. One can also consider something more general where $\mathcal{R}$ is a cocycle rather than like a quasitriangular structure. The above definition is stronger but is the one that applies to our bosonisations. From Proposition 2.3 and Lemma 2.5 we have clearly:

Corollary 2.7. *If B is a $*$-braided group acted upon unitarily as in (19) by a real-quasitriangular $*$-quantum group H, then its bosonisation $B{>\!\!\!\triangleleft} H$ is a quasi-$*$ Hopf algebra with B, H as sub $*$-algebras.*

Finally, we prove a related and somewhat harder theorem which we will also use in a later section (as a quantum Wick rotation between Euclidean and Minkowski–Poincaré groups.) Namely, we consider how the bosonisation construction responds to twisting under a cocycle. It is clear that if B is an H-module algebra (an algebra in the category of H-modules) and we twist H by a 2-cocycle χ as in (12) then we must also 'twist' the algebra of B in a certain way if we want it to remain covariant. Likewise, if we have a coalgebra in the category then we have to 'twist' that too if we want to stay in the category of H-modules.

Theorem 2.8. *If B is a braided group in the category of H-modules, and χ a 2-cocycle for H, then B_χ defined by*

$$b\cdot_\chi c=\cdot\circ\chi^{-1}\triangleright(b\otimes c),\qquad \underline{\Delta}_\chi b=\chi\triangleright\underline{\Delta}b,\qquad \underline{\epsilon}_\chi=\underline{\epsilon},\qquad \underline{S}_\chi=\underline{S}\tag{20}$$

is a braided group in the category of H_χ-modules. If B is only a braided-(bi)algebra then so is B_χ.

Proof. As explained in [28], for example, we know that if B is a (co)algebra covariant under H then B_χ is a (co)algebra covariant under H_χ. We have to check that this twisted algebra and coalgebra still fit together to form a braided group in the braided category of H_χ-modules. Thus

$$\begin{aligned}
&\underline{\Delta}_\chi(b\cdot_\chi c)\\
&=\underline{\Delta}((\chi^{-(1)}\triangleright b)(\chi^{-(2)}\triangleright c))\\
&=\chi^{(1)}\triangleright((\chi^{-(1)}\triangleright b)_{\underline{(1)}}\mathcal{R}^{(2)}\triangleright(\chi^{-(2)}\triangleright c)_{\underline{(1)}})\\
&\quad\otimes\chi^{(2)}\triangleright((\mathcal{R}^{(1)}\triangleright(\chi^{-(1)}\triangleright b)_{\underline{(2)}})(\chi^{-(2)}\triangleright c)_{\underline{(2)}})\\
&=(\chi^{(1)}{}_{(1)}\chi^{-(1)}{}_{(1)}\triangleright b_{\underline{(1)}})(\chi^{(1)}{}_{(2)}\mathcal{R}^{(2)}\chi^{-(2)}{}_{(1)}\triangleright c_{\underline{(1)}})\\
&\quad\otimes(\chi^{(2)}{}_{(1)}\mathcal{R}^{(1)}\chi^{-(1)}{}_{(2)}\triangleright b_{\underline{(2)}})(\chi^{(2)}{}_{(2)}\chi^{-(2)}{}_{(2)}\triangleright c_{\underline{(2)}}),\\
&(\underline{\Delta}_\chi b)\cdot_\chi(\underline{\Delta}_\chi c)\\
&=(\chi^{-(1)}\chi^{(1)}\triangleright b_{\underline{(1)}})(\chi^{-(2)}\mathcal{R}_\chi{}^{(2)}\chi'^{(1)}\triangleright c_{\underline{(1)}})\\
&\quad\otimes(\chi'^{-(1)}\mathcal{R}_\chi{}^{(1)}\chi^{(2)}\triangleright b_{\underline{(2)}})(\chi'^{-(2)}\chi'^{(2)}\triangleright c_{\underline{(2)}}),
\end{aligned}$$

using the definitions of the product and coproduct of B_χ. In the second equality we use that B itself is a braided group in the category with braiding defined via $\mathcal{R}$, and for the third we use covariance of its product under H. We seek equality for all $b, c \in B$ with the lower expression, which is the braided tensor product of $\underline{\Delta}_\chi$ applied to b, c in the category with braiding defined by $\mathcal{R}_\chi = \chi_{21}\mathcal{R}\chi^{-1}$. Equality holds in view of the identity

$$((\Delta\otimes\Delta)\chi)\mathcal{R}_{32}\Delta_{H\otimes H}\chi^{-1}=\chi_{12}^{-1}\chi_{34}^{-1}\chi_{23}\mathcal{R}_{32}\chi_{32}^{-1}\chi_{13}\chi_{24},$$

which follows from repeated use of the cocycle condition (11) and the quasitriangularity of $\mathcal{R}$. It is clear that the unit and counit are not affected by the twisting. Hence we have a braided bialgebra B_χ. If B has a braided antipode then it is clear that the same map provides a braided antipode for B_χ. This is because χ acts when making the coproduct of B_χ and χ^{-1} acts when making the product, and $\underline{S}$ is an intertwiner for the action, so that χ^{-1}, χ cancel. □

Theorem 2.8 fits together with the twisting (12) of quantum groups to tell us that the process of twisting and the process of bosonisation commute. In categorical terms the reason is that the category of H_χ-covariant representations of B_χ is equivalent to the category of H-covariant representations of B, the equivalence respecting tensor products up to χ. But the first category is isomorphic to the category of $B_\chi \rtimes H_\chi$-modules and the second to that of $B{>\!\!\!\triangleleft} H$-modules. These are therefore equivalent up to χ. This means by Tannaka–Krein arguments [13] that these two bosonisations are twisting-equivalent as Hopf algebras.

Proposition 2.9. *In the setting of Theorem 2.8, we have*

$$B_\chi{>\!\!\!\triangleleft} H_\chi \cong (B{>\!\!\!\triangleleft} H)_\chi,$$

where on the right-hand side we view χ as a 2-cocycle for $B{>\!\!\!\triangleleft} H$, i.e., the bosonisation of the twisted braided group is the twisted quantum group of its bosonisation.

Proof. The categorical argument sketched makes this a corollary of Theorem 2.8. Here we show directly that the required isomorphism is provided by $\theta : (B{>\!\!\!\triangleleft} H)_\chi \to B_\chi{>\!\!\!\triangleleft} H_\chi$, where $\theta(bh) = (\chi^{(1)}\triangleright b)\chi^{(2)}h$. This is the identity on the H_χ sub-Hopf algebra, as it should be since both sides contain this. In addition

$$\begin{aligned}
\theta(bc) &= (\chi^{(1)}\triangleright(bc))\chi^{(2)} = ({\chi^{(1)}}_{(1)}\triangleright b)({\chi^{(1)}}_{(2)}\triangleright c)\chi^{(2)}\\
&= (\chi'^{(1)}\chi^{(1)}\triangleright b)(\chi'^{(2)}\chi''^{(1)}{\chi^{(2)}}_{(1)}\triangleright c)\chi''^{(2)}{\chi^{(2)}}_{(2)}\\
&= (\chi^{(1)}\triangleright b)\cdot_\chi(\chi''^{(1)}{\chi^{(2)}}_{(1)}\triangleright c)\chi''^{(2)}{\chi^{(2)}}_{(2)}\\
&= (\chi^{(1)}\triangleright b)\chi^{(2)}(\chi'^{(1)}\triangleright c)\chi'^{(2)} = \theta(b)\theta(c),
\end{aligned}$$

where we use the H-covariance of B for the second equality, the 2-cocycle condition (11) for χ for the third, the definition of the product of B_χ for the fourth, and finally the cross relations in $B_\chi{>\!\!\!\triangleleft} H_\chi$ from (9). We also verify the cross relations as

$$\theta(h)\theta(b) = h(\chi^{(1)}\triangleright b)\chi^{(2)} = (\chi^{(1)}h_{(1)}\triangleright b)\chi^{(2)}h_{(2)} = \theta(h_{(1)}\triangleright b)\theta(h_{(2)}),$$

where we use the relations in $B_\chi{>\!\!\!\triangleleft} H_\chi$ again. Hence θ is an algebra homomorphism.

For the coproduct of $B_\chi{>\!\!\!\triangleleft} H_\chi$ we compute from (9)

$$\begin{aligned}
\Delta(\theta(b)) &= (\chi'^{(1)}\triangleright(\chi^{(1)}\triangleright b)_{\underline{(1)}}){\mathcal{R}_\chi}^{(2)}\chi''^{(1)}{\chi^{(2)}}_{(1)}\chi^{-(1)}\\
&\quad\otimes({\mathcal{R}_\chi}^{(1)}\chi'^{(2)}\triangleright(\chi^{(1)}\triangleright b)_{\underline{(2)}})\chi''^{(2)}{\chi^{(2)}}_{(2)}\chi^{-(2)}\\
&= (\chi'^{(1)}{\chi^{(1)}}_{(1)}\triangleright b_{\underline{(1)}}){\mathcal{R}_\chi}^{(2)}\chi''^{(1)}{\chi^{(2)}}_{(1)}\chi^{-(1)}\\
&\quad\otimes({\mathcal{R}_\chi}^{(1)}\chi'^{(2)}{\chi^{(1)}}_{(2)}\triangleright b_{\underline{(2)}})\chi''^{(2)}{\chi^{(2)}}_{(2)}\chi^{-(2)},\\
\chi((\theta\otimes\theta)\circ\Delta b)\chi^{-1} &= \chi'^{(1)}(\chi^{(1)}\triangleright b_{\underline{(1)}})\chi^{(2)}\mathcal{R}^{(2)}\chi'^{-(1)}\\
&\quad\otimes\chi'^{(2)}(\chi''^{(1)}\mathcal{R}^{(1)}\triangleright b_{\underline{(2)}})\chi''^{(2)}\chi'^{-(2)}\\
&= (\chi^{(1)}{\chi'^{(1)}}_{(1)}\triangleright b_{\underline{(1)}})\chi^{(2)}{\chi'^{(1)}}_{(2)}\mathcal{R}^{(2)}\chi^{-(1)}\\
&\quad\otimes(\chi''^{(1)}{\chi'^{(2)}}_{(1)}\mathcal{R}^{(1)}\triangleright b_{\underline{(2)}})\chi''^{(2)}{\chi'^{(2)}}_{(2)}\chi^{-(2)},
\end{aligned}$$

where we used the twisted braided coproduct of B_χ and the twisted quasitriangular structure of H_χ for the first equality, and H-covariance of the braided coproduct of B for the second. We seek equality with the lower expression, computed using the relations of $B_\chi{>\!\!\!\triangleleft} H_\chi$ obtained from (9). Equality holds for all $b\in B$ in view of the identity

$$\chi_{23}\mathcal{R}_{32}\chi_{32}^{-1}\chi_{13}\chi_{24}(\Delta_{H\otimes H}\chi) = \chi_{12}\chi_{34}((\Delta\otimes\Delta)\chi)\mathcal{R}_{32}$$

as in the proof of Theorem 2.8. That the unit and counit map over correctly is immediate, after which it follows that the antipodes also map over. Hence θ is an isomorphism of Hopf algebras. □

We will use this in Section 3.2. For completeness, we also discuss the interaction of $*$ with twisting.

Proposition 2.10. *If H is a Hopf $*$-algebra and χ is a 2-cocycle for it which is real in the sense $(S\otimes S)(\chi^{*\otimes *})=\chi_{21}$, then:*

(i) *H_χ is a Hopf $*$-algebra. If H is (anti) real-quasitriangular then so is H_χ.*

(ii) *If B is a $*$-braided-group in the category of H-modules with H real-quasitriangular and the action 'unitary' in the sense (19), then B_χ is a $*$-braided group in the category of H_χ-modules.*

(iii) *In this case, $B_\chi \rtimes H_\chi$ is a quasi-$*$ Hopf algebra by Corollary 2.7. The required $*$-structures on H_χ and B_χ, respectively, are*

$$*_\chi(h)=(S^{-1}U)h^*S^{-1}U^{-1},\qquad *_\chi(b)=(S^{-1}U)S^{-2}U^{-1}\triangleright b^*.$$

Proof. The first part belongs to the theory of twisting of quantum groups, mentioned in Section 1.1. Since it does not seem to be discussed previously, we include a proof. From the stated 'reality' assumption for χ we see that $U^*=S^{-2}U$ and hence that $S^{-1}U$ is real. This implies that $(*_\chi)^2=\mathrm{id}$, making H_χ into a $*$-algebra. Moreover, from the form of S_χ in (12) we see that $S_\chi\circ *_\chi(h)=U(S((S^{-1}U)h^*S^{-1}U^{-1}))U^{-1}=S(h^*)$ so that $(S_\chi\circ *_\chi)^2=\mathrm{id}$ as required. More non-trivial is the coproduct. However, it was shown in [41, Lemma 2.2] that

$$\Delta U^{-1}=(S\otimes S)(\chi_{21})(U^{-1}\otimes U^{-1})\chi \tag{21}$$

from which we conclude that $\Delta(S^{-1}U^{-1})=\chi^{*\otimes *}(S^{-1}U^{-1}\otimes S^{-1}U^{-1})\chi$ under our reality assumption for χ. This implies at once

$$\begin{aligned}(*_\chi\otimes *_\chi)(\Delta_\chi h)&=(S^{-1}U)\chi^{-(1)*}h^*{}_{(1)}\chi^{(1)*}S^{-1}U^{-1}\\&\quad\otimes(S^{-1}U)\chi^{-(2)*}h^*{}_{(2)}\chi^{(2)*}S^{-1}U^{-1}\\&=\chi^{(1)}(S^{-1}U)_{(1)}h^*{}_{(1)}(S^{-1}U^{-1})_{(1)}\chi^{-(1)}\\&\quad\otimes\chi^{(2)}(S^{-1}U)_{(2)}h^*{}_{(2)}(S^{-1}U^{-1})_{(2)}\chi^{-(2)}\\&=\Delta_\chi\circ *_\chi(h)\end{aligned}$$

as required. Finally, we check that if H is real-quasitriangular, then

$$\begin{aligned}(*_\chi\otimes *_\chi)(\mathcal{R}_\chi)&=(S^{-1}U\otimes S^{-1}U)\chi^{-1*\otimes *}\mathcal{R}_{21}\chi_{21}^{*\otimes *}(S^{-1}U^{-1}\otimes S^{-1}U^{-1})\\&=\chi(\Delta S^{-1}U)\mathcal{R}_{21}(\tau\circ\Delta S^{-1}U^{-1})\chi_{21}^{-1}=(\mathcal{R}_\chi)_{21}\end{aligned}$$

using the result on $\Delta S^{-1}U^{-1}$ and the quasitriangularity assumption. Similarly if $\mathcal{R}$ is anti-real.

For the second part, we suppose B is a $*$-algebra in the category of modules of a Hopf $*$-algebra H, and that the latter acts as in (19). We define $*_\chi$ on B as stated. Then $(*_\chi)^2(b) = (S^{-1}U)S^{-2}U^{-1} \triangleright ((S^{-1}U)S^{-2}U^{-1} \triangleright b^*)^* = b$ as required, using (19). Moreover, if we write for brevity $\gamma \equiv (S^{-1}U)S^{-2}U^{-1}$ then our key property (21) tells us that

$$\Delta\gamma = \chi^{-1}(\gamma\otimes\gamma)(S^{-2}\otimes S^{-2})(\chi). \tag{22}$$

Then when we consider the algebra B_χ with the twisted product $\cdot_\chi$ as in Theorem 2.8, we will have

$$\begin{aligned}
*_\chi((b\cdot_\chi c)) &= \gamma\triangleright((\chi^{-(2)}\triangleright c)^*(\chi^{-(1)}\triangleright b)^*)\\
&= (\gamma_{(1)}(S\chi^{-(2)})^*\triangleright c^*)(\gamma_{(2)}(S\chi^{-(1)})^*\triangleright b^*)\\
&= (\chi^{-(1)}\gamma\triangleright c^*)(\chi^{-(2)}\gamma\triangleright b^*) = (*_\chi(c))\cdot_\chi(*_\chi(b))
\end{aligned}$$

using our reality assumption on χ and (22). Hence B_χ is a $*$-algebra under $*_\chi$. Likewise, suppose that B is an anti-$*$-coalgebra in the sense of the coproduct axiom in (4) and the action, and covariant under the Hopf $*$-algebra H as before. Then the twisted coproduct $\underline{\Delta}_\chi$ as in Theorem 2.8 obeys

$$\begin{aligned}
(*_\chi\otimes *_\chi)\underline{\Delta}_\chi(b) &= \gamma\triangleright(\chi^{(1)}\triangleright b_{\underline{(1)}})^*\otimes\gamma\triangleright(\chi^{(2)}\triangleright b_{\underline{(2)}})^*\\
&= \gamma(S\chi^{(1)})^*\triangleright b^*{}_{\underline{(2)}}\otimes\gamma(S\chi^{(2)})^*\triangleright b^*{}_{\underline{(1)}}\\
&= \chi^{(2)}\gamma_{(2)}\triangleright b^*{}_{\underline{(2)}}\otimes\chi^{(1)}\gamma_{(1)}\triangleright b^*{}_{\underline{(1)}} = \tau\circ\underline{\Delta}_\chi\circ *_\chi(b)
\end{aligned}$$

as required.

So in particular, if B is a $*$-braided group then so is B_χ. Moreover we know from Theorem 2.8 that H_χ acts on it. We check that its action obeys condition (19). Thus

$$\begin{aligned}
*_\chi(h\triangleright b) &= \gamma(Sh)^*\triangleright b^* = S^{-1}(U^{-1}(S^{-1}U)h^*(S^{-1}U^{-1})U)\triangleright *_\chi(b)\\
&= S_\chi^{-1}(*_\chi(h))\triangleright *_\chi(b)
\end{aligned}$$

as required. Since we have also seen that H_χ is real-quasitriangular when H is, we are in the situation of Corollary 2.7 and conclude that the bosonisation $B_\chi \rtimes\!\!\!\blacktriangleleft H_\chi$ is a quasi-$*$ Hopf algebra for the $*$-structures as stated. It is related to $B\rtimes\!\!\!\blacktriangleleft H$ by a theory of twisting of quasi-$*$ Hopf algebras. □

3. Poincaré quantum enveloping algebras of braided linear spaces

In this section we specialise our preceding results to the case where B is a braided vector space $V(R', R)$ as introduced in [9]. The data are two matrices $R', R \in M_n\otimes M_n$ obeying the equations

$$\begin{aligned}
R'_{12}R_{13}R_{23} &= R_{23}R_{23}R'_{12}, \qquad & R_{12}R_{13}R'_{23} &= R'_{23}R_{13}R_{12},\\
R_{12}R_{13}R_{23} &= R_{23}R_{13}R_{12}, \qquad & (PR+1)(PR'-1) &= 0,
\end{aligned} \tag{23}$$

where P is the usual permutation matrix. Then V is defined with generators $\boldsymbol{p} = \{p^i\}$ and the relations $\boldsymbol{p}_1\boldsymbol{p}_2 = R'\boldsymbol{p}_2\boldsymbol{p}_1$, which is *not* the usual Zamolodchikov or exchange algebra since we do not assume that R' obeys the QYBE (though it often does in practice). Rather, the QYBE applies to R, which we use in defining the braiding $\Psi(\boldsymbol{p}_1 \otimes \boldsymbol{p}_2) = R\boldsymbol{p}_2 \otimes \boldsymbol{p}_1$. We take braided coproduct $\underline{\Delta}\boldsymbol{p} = \boldsymbol{p} \otimes 1 + 1 \otimes \boldsymbol{p}$. This can be expressed as the addition of braided co-cordinates, see [9,22] where the construction was introduced and applied, respectively.

We assume further that the matrix R is *regular* in the sense [21] that we can build from the usual quantum matrix bialgebra $A(\lambda R)$ [32] a quantum group A by adding further relations, such that the canonical (dual) quasitriangular structure on $A(\lambda R)$ introduced in [13] descends to A. Here λ is a constant which does not enter into the relations of $A(\lambda R)$ but does affect its dual-quasitriangular structure. It is the *quantum group normalisation constant* introduced in this way in [21]. We also make a covariance assumption [9] that the matrix R' is compatible with the quantum group A in the sense $R'\boldsymbol{t}_1\boldsymbol{t}_2 = \boldsymbol{t}_2\boldsymbol{t}_1R'$. This is generally true, for example if PR' is a function of PR.

Finally, we assume that there is a quasitriangular Hopf algebra H dually paired with A. In this case, define the H-valued matrices

$$\boldsymbol{l}^+ = \langle \mathrm{id} \otimes \boldsymbol{t}, \mathcal{R}\rangle, \qquad \boldsymbol{l}^- = \langle \boldsymbol{t} \otimes \mathrm{id}, \mathcal{R}^{-1}\rangle, \tag{24}$$

which necessarily obey (among other relations) the quadratic relations in [32]. We assume that the elements of these matrices generate H at least over formal powerseries in a deformation parameter. All the above conditions are satisfied of course for the standard deformations $U_q(g)$ associated to a complex semisimple Lie algebra, but it is necessary for us not to be tied to this case to cover other interesting examples as well. This method to obtain $\boldsymbol{l}^\pm$ from $\mathcal{R}$ was used [48] for $U_q(su_3)$.

This describes all the data for the bosonisation construction, in R-matrix form. The Poincaré enveloping algebras were then constructed under these assumptions in [9] by bosonisation (9), as well as the their dual quantum groups (which we do not discuss explicitly) by cobosonisation (10). To H we have to add a central primitive generator ξ with quasitriangular structure $\lambda^{-\xi\otimes\xi}$ which we multiply into the quasitriangular structure $\mathcal{R}$ of H above. This is the quantum group $\tilde{H}$ which we actually use. The braided vectors $V(R', R)$ live in its category of modules by the action

$$\boldsymbol{l}_1^+ \triangleright \boldsymbol{p}_2 = \lambda^{-1}R_{21}^{-1}\boldsymbol{p}_2, \qquad \boldsymbol{l}_1^- \triangleright \boldsymbol{p}_2 = \lambda R\boldsymbol{p}_2, \qquad \lambda^\xi \triangleright \boldsymbol{p} = \lambda^{-1}\boldsymbol{p}. \tag{25}$$

In [9] we emphasised the right coaction of the dilatonic extension of A (the p^i transform as a quantum vector); the above is nothing more than an evaluation of $\boldsymbol{l}^\pm, \xi$ against that coaction. The *inhomogeneous quantum group* $V(R', R) \rtimes\!\!\!\cdot\, \tilde{H}$ of the braided linear space is then constructed from (9) as generated by $\boldsymbol{p}, \boldsymbol{l}^\pm, \xi$ and with cross relations, coproduct and antipode [9]

$$\begin{aligned} &\boldsymbol{l}_1^+\boldsymbol{p}_2 = \lambda^{-1}R_{21}^{-1}\boldsymbol{p}_2\boldsymbol{l}_1^+, \quad \boldsymbol{l}_1^-\boldsymbol{p}_2 = \lambda R\boldsymbol{p}_2\boldsymbol{l}_1^-, \quad \lambda^\xi\boldsymbol{p} = \lambda^{-1}\boldsymbol{p}\lambda^\xi, \\ &\Delta\boldsymbol{p} = \boldsymbol{p} \otimes 1 + \lambda^\xi\boldsymbol{l}^- \otimes \boldsymbol{p}, \quad \epsilon\boldsymbol{p} = 0, \quad S\boldsymbol{p} = -\lambda^{-\xi}(S\boldsymbol{l}^-)\boldsymbol{p}. \end{aligned} \tag{26}$$

It should be clear from [9] that the reason why there is only $\boldsymbol{l}^-$ and not $\boldsymbol{l}^+$ in $\Delta\boldsymbol{p}$ is that we used $\mathcal{R}$ and not $\mathcal{R}_{21}^{-1}$ in (9).

Proposition 3.1. *The Hopf algebra $V(R', R){>\!\!\!\triangleleft}\,\tilde{H}$ in the setting above has a second 'conjugate' coproduct and antipode from Proposition 2.3, namely*

$$\bar{\Delta}\boldsymbol{p} = \boldsymbol{p}\otimes 1 + \lambda^{-\xi}\boldsymbol{l}^{+}\otimes\boldsymbol{p}, \qquad \bar{S}\boldsymbol{p} = -\lambda^{\xi}(S\boldsymbol{l}^{+})\boldsymbol{p}. \tag{27}$$

Proof. The computation is exactly the same in form as in [9] and follows at once from (24), (25) and (17). For example, the 'conjugate' coproduct of $\boldsymbol{p}$ is computed as $\boldsymbol{p}\otimes 1 + \mathcal{R}^{-(1)}\otimes\mathcal{R}^{-(2)}\triangleright\boldsymbol{p}$ which yields $\boldsymbol{l}^{+}$ because the action of $\mathcal{R}^{-(2)}$ is by evaluation against the coaction $p^{i}\to p^{a}St^{i}{}_{a}$, which we compute from (24) and $(\mathrm{id}\otimes S)(\mathcal{R}^{-1}) = \mathcal{R}$. One may check directly that the previous coproduct and the new 'conjugate' one are related by conjugation by $\mathcal{R}^{-1}$, as they must be from Proposition 2.3. □

To proceed further and obtain a quasi-$*$ Hopf algebra, we need to suppose that $V(R', R)$ is a $*$-braided group. This question was analysed in [8] and there are several possibilities. The simplest, which covers q-Euclidean spaces, is to assume that R is of real type I in the sense $\bar{R} = R^{\dagger\otimes\dagger}$ with real quantum group normalisation constant λ, and that there is a covariant *quantum metric* η_{ij}. The construction of such quantum metrics from braided geometry has been covered in [23]. We require a tensor that is quantum group invariant and obeys various identities with respect to R, R' such as to make $V(R', R)$ isomorphic to the braided covectors $V\check{}(R', R)$ as braided groups in the category generated by R. The braided covectors are defined in [9] with generators $\{p_i\}$ where the indices are down, and corresponding relations like the above but R', R acting from the left on the covector generators; we require that $p_i = \eta_{ia}p^{a}$ effects an isomorphism, cf. [26]. One may deduce various useful identities, among them (coming from invariance) the identities [24]

$$\eta_{ia}R^{-1a}{}_{j}{}^{k}{}_{l} = \lambda^{2}R^{a}{}_{i}{}^{k}{}_{l}\eta_{aj}, \qquad \eta_{ka}R^{i}{}_{j}{}^{a}{}_{l} = \lambda^{-2}R^{-1i}{}_{j}{}^{a}{}_{k}\eta_{al}, \tag{28}$$

which we particularly need below. We use conventions in which η^{ij} is the inverse transpose of η, and assume further the reality condition $\overline{\eta_{ij}} = \eta^{ji}$. Then we can take $p^{i*} = p_i$ as explained in [8].

We also need that H is a real-quasitriangular Hopf $*$-algebra. This will generally follow from the other conditions as long as R is of real type. For then the $*$-structure of A can typically be taken in the compact from $t^{i}{}_{j}{}^{*} = St^{j}{}_{i}$ as in [10]. The dual-quasitriangular structure of A is then necessarily of real type.

Proposition 3.2. *Under the reality assumption on R and given a quantum metric, the inhomogeneous quantum group $V(R', R){>\!\!\!\triangleleft}\,\tilde{H}$ becomes a quasi-$*$ Hopf algebra with*

$$l^{\pm i}{}_{j}{}^{*} = Sl^{\mp j}{}_{i}, \quad p^{i*} = p_i, \quad \xi^{*} = -\xi. \tag{29}$$

We regard the p_i as linear combinations of the p^{i}, just with 'lowered indices'.

Proof. The form of $*$ on the p^{i} is from [8]. The form on $l^{\pm}$ is the standard one in the case of $U_q(g)$ but is deduced in our more general setting above from (24) and the reality type type of $\mathcal{R}$. This was explained in [25]. We also need that action [25] is 'unitary', which is easily checked from the reality type of R. Thus

$$\begin{aligned}(l^{+i}{}_j \triangleright p^k)^* &= \overline{\lambda^{-1} R^{-1k}{}_b{}^i{}_j p^{b*}} = \lambda^{-1} R^{-1j}{}_i{}^b{}_k \eta_{ba} p^a\\ &= \lambda \eta_{kb} R^j{}_i{}^b{}_a p^a = l^{-j}{}_i \triangleright \eta_{kb} p^b = (Sl^{+i}{}_j)^* \triangleright p^{k*}\end{aligned}$$

and similarly for the action of $\boldsymbol{l}^-$. We use Eqs. (28). We then deduce that we have a quasi-$*$ Hopf algebra from Corollary 2.7. □

Since the p_i are equally good generators in our setting (or if we just want to work with lower indices in any case) there is nothing stopping us giving the inhomogeneous quantum group relations in lowered-index form. We can proceed from the above, using the metric relations (28) or directly by bosonisation of $V^\vee(R', R)$. The results are the same except for a difference in sign of ξ which can be absorbed by a redefinition. In the first point of view, the result is

$$\begin{aligned}&\boldsymbol{p}_1\boldsymbol{p}_2 = \boldsymbol{p}_2\boldsymbol{p}_1 R', \qquad \boldsymbol{l}_1^+\boldsymbol{p}_2 = \lambda \boldsymbol{p}_2 R_{21}\boldsymbol{l}_1^+,\\ &\boldsymbol{l}_1^-\boldsymbol{p}_2 = \lambda^{-1}\boldsymbol{p}_2 R^{-1}\boldsymbol{l}_1^-, \qquad \lambda^\xi \boldsymbol{p} = \lambda^{-1}\boldsymbol{p}\lambda^\xi,\\ &\Delta p_i = p_i \otimes 1 + \lambda^\xi S l^{-a}{}_i \otimes p_a, \qquad \epsilon \boldsymbol{p} = 0,\\ &S p_i = -\lambda^{-\xi}(S^2 l^{-a}{}_i) p_a,\\ &\bar{\Delta} p_i = p_i \otimes 1 + \lambda^{-\xi} S l^{+a}{}_i \otimes p_a,\\ &\bar{S} p_i = -\lambda^{\xi}(S^2 l^{+a}{}_i) p_a,\end{aligned} \tag{30}$$

where now $\boldsymbol{p} = \{p_i\}$.

The above theory includes, for example, the Euclidean group of motions $\mathbb{R}_q^n \rtimes U_q(\widetilde{so_n})$ where $U_q(\widetilde{so_n})$ is a central extension of the standard q-deformed enveloping algebra, which we take in FRT from [32] and with Drinfelds's quasitriangular structure [1]. The appropriate $\mathbb{R}_q^n$ are the quantum planes in [32] for suitable R'. One of the major results in [9] was that the Poincaré quantum groups obtained in this way automatically (co)act on the space–time braided covectors x_i. As also explained in [9] and developed fully in its sequel [22], the coaction by the braided addition becomes by evaluation an action of the Poincaré enveloping algebra momentum generators p^i on space–time by braided differentiation. Since we have two coproducts, we obviously have two such actions by differentials and 'conjugate' differentials. These generate the fundamental and conjugate fundamental representations from Corollaries 2.2 and 2.4, and are studied further in Sections 4 and 5. Let us note that this Hopf algebra $\mathbb{R}_q^n \rtimes U_q(\widetilde{so_n})$ and the q-Euclidean space on which it acts have recently been studied by rather more explicit means in [49].

3.1. Spinorial q-Euclidean–Poincaré enveloping algebra

There are many classes of linear braided groups, much beyond the usual quantum planes associated with representations of standard quantum group deformations such as $\mathbb{R}_q^n \rtimes U_q(\widetilde{so_n})$. In this section and the next, we specialise to a class in which the generators p_i above are replaced by a martix of generators $\boldsymbol{p} = \{p^{i_0}{}_{i_1}\}$ say. We are still considering them as an additive braided group but adopt a notation in which $\boldsymbol{p}$ refers to a matrix, with the usual notational rules. For example, the matrix bialgebras $A(R)$ have such a coaddition

[50], as do rectangular quantum matrices $A(R:S)$ in [51], whenever R, S are q-Hecke solutions of the QYBE (so that the corresponding braiding has eigenvalues $q, -q^{-1}$).

In particular, we focus here on the 'rectangular quantum matrix' algebra $\bar{A}(R) = A(R_{21} : R)$ proposed in the 2×2 case as a matrix or 'twistor' description of four-dimensional q-Euclidean space in [28]. It has relations and braid statistics.

$$R_{21}p_1p_2 = p_2p_1R, \qquad p'_1p_2 = Rp_2p'_1R \tag{31}$$

under which it forms a braided group with $\underline{\Delta}\boldsymbol{p} = \boldsymbol{p}\otimes 1 + 1\otimes \boldsymbol{p}$ as before. Clearly we can write $p^{i_0}{}_{i_1} = p_I$ as a covector with multiindex $I = (i_0, i_1)$, and in this way write this braided linear space as $V\check{}(\boldsymbol{R}', \boldsymbol{R})$ for suitable 'multiindex' $\boldsymbol{R}', \boldsymbol{R}$ given in [28]. So the difference is purely notational.

One can then follow the preceding section with the matrices $\boldsymbol{R}', \boldsymbol{R}$ and quantum group generators $l^{\pm I}{}_J$ using the setting there. Alternatively, which we do in the present section, we can take a slightly different quantum group as the 'background symmetry' with respect to which we bosonise. Namely, we take in the role of H in the preceding section a quantum group $H \otimes H$, where this time H is a quasitriangular Hopf algebra dual to a dual-quasitriangular Hopf algebra A obtained from $A(\lambda^{1/2}R)$. We take the tensor product quasitriangular Hopf algebra structure. This quantum group is related to the one in the preceding section by the realisation

$$l^{\pm I}{}_J = (S^{-1}l^{\pm j_0}{}_{i_0})m^{\pm i_1}{}_{j_1}, \tag{32}$$

where $\boldsymbol{l}^{\pm}$ and $\boldsymbol{m}^{\pm}$ denote the matrix generators of the two copies of H. It is still the case that $\bar{A}(R)$ lives in the braided category of a dilatonic extension of $H \otimes H$, so we construct this and proceed directly by bosonisation. This formulation has been explanied in [28] and we already know from there that the space–time co-ordinates become a module algebra under the space-time rotation. We use the same action on the lowered-index momentum generators $p_I = p^{i_0}{}_{i_1}$, regarded now as a matrix, namely [28]

$$\begin{aligned} &\boldsymbol{l}_1^+ \triangleright \boldsymbol{p}_2 = \lambda^{-1/2}R_{21}^{-1}\boldsymbol{p}_2, \quad \boldsymbol{l}_1^- \triangleright \boldsymbol{p}_2 = \lambda^{1/2}R\boldsymbol{p}_2\\ &\boldsymbol{m}_1^+ \triangleright \boldsymbol{p}_2 = \boldsymbol{p}_2\lambda^{1/2}R_{21}, \qquad \boldsymbol{m}_1^- \triangleright \boldsymbol{p}_2 = \boldsymbol{p}_2\lambda^{-1/2}R^{-1}. \end{aligned} \tag{33}$$

We add a dilaton ξ as before, with λ the quantum group normalisation constant for $\boldsymbol{R}$, which is the square of that for R.

Proposition 3.3. *The inhomogeneous quantum group $\bar{A}(R)\rtimes\!\!\!\lhd(H\widetilde{\otimes} H)$ constructed from (9) has cross relations, coproduct and antipode*

$$\begin{aligned} &\boldsymbol{l}_1^+\boldsymbol{p}_2 = \lambda^{-1/2}R_{21}^{-1}\boldsymbol{p}_2\boldsymbol{l}_1^+, \quad \boldsymbol{l}_1^-\boldsymbol{p}_2 = \lambda^{1/2}R\boldsymbol{p}_2\boldsymbol{l}_1^-,\\ &\boldsymbol{m}_1^+\boldsymbol{p}_2 = \boldsymbol{p}_2\lambda^{1/2}R_{21}\boldsymbol{m}_1^+, \quad \boldsymbol{m}_1^-\boldsymbol{p}_2 = \boldsymbol{p}_2\lambda^{-1/2}R^{-1}\boldsymbol{m}_1^-, \quad \lambda^{\xi}\boldsymbol{p} = \lambda^{-1}\boldsymbol{p}\lambda^{\xi}\\ &\Delta\boldsymbol{p} = \boldsymbol{p}\otimes 1 + \lambda^{\xi}(\boldsymbol{l}^-(\ )S\boldsymbol{m}^-)\otimes\boldsymbol{p}, \quad \epsilon\boldsymbol{p} = 0,\\ &S\boldsymbol{p} = -\lambda^{-\xi}S(\boldsymbol{l}^-(\ )S\boldsymbol{m}^-)\boldsymbol{p}, \end{aligned} \tag{34}$$

where $\boldsymbol{l}^-(\ )S\boldsymbol{m}^-$ has a space for the matrix indices of $\boldsymbol{p}$ to be inserted. The second 'conjugate' coproduct and antipode from Proposition 2.3 *are*

$$\bar{\Delta} p = p \otimes 1 + \lambda^{-\xi} l^{+}(\ ) S m^{+} \otimes p, \qquad \bar{S} p = -\lambda^{\xi} S(l^{+}(\ ) S m^{+}) p. \tag{35}$$

Proof. The semidirect product (9) gives the cross relations as before. We read them off from (33) because of the matrix form of the coproducts of $l^{\pm}, m^{\pm}$. For the coproduct and conjugate coproduct we evaluate against the composite $\mathcal{R}$ (which is the tensor product of one for each copy). The computation follows the same line as in [9] and Proposition 3.1, for each copy separately, giving the result. □

This has been announced in [11]. It is dual by Lemma 2.1 to the spinorial Poincaré enveloping algebra computed in [28]. Next we consider $*$-structures. One can take various $*$-structures on $\bar{A}(R)$ (including one as a Hermitian matrix) but we concentrate here on 'unitary type' $*$-structures defined according to the twisting theory in [28] by having the same form as on the generators t of the quantum group A obtained from $A(\lambda^{1/2}R)$. We suppose this has the explicit form $t^i{}_j{}^* = \epsilon_{ai} t^a{}_b \epsilon^{bj}$ say, where ϵ^{ij} is quantum group invariant and ϵ_{ij} the transposed inverse. For $H \otimes H$, the $*$-structure we need according to the theory in [28] is not quite the tensor product one, but has an extra automorphism by S^{-2} in the first copy. We assume as before that H is a real-quasitriangular Hopf algebra dual to A. For both these assumptions we assume that R is of real type I in the sense $\bar{R} = R^{\dagger \otimes \dagger}$ and that $\lambda^{1/2}$ is real.

Proposition 3.4. *Under the reality assumption on R and given a suitable ϵ as above, the inhomogeneous quantum group $\bar{A}(R) \rtimes H \widetilde{\otimes} H$ becomes a quasi-$*$ Hopf algebra with*

$$l^{\pm i}{}_j{}^* = S^{-1} l^{\mp j}{}_i, \quad m^{\pm i}{}_j{}^* = S m^{\mp j}{}_i, \quad p^i{}_j{}^* = \epsilon_{ai} p^a{}_b \epsilon^{bj}, \quad \xi^* = -\xi. \tag{36}$$

Proof. The quantum group $H \otimes H$ is real-quasitriangular since each factor is. The $*$-structure on $m^{\pm}$ is as in the preceding section. On the $l^{\pm}$ we have the extra automorphism S^{-2}. We also need action (33) to be unitary in the sense (19). Thus

$$\begin{aligned}
(m^{+i}{}_j \triangleright p^k{}_l)^* &= (p^k{}_a \lambda^{1/2} R^a{}_l{}^i{}_j)^* = \lambda^{1/2} R^j{}_i{}^l{}_a \epsilon_{bk} p^b{}_c \epsilon^{ca} = \lambda^{-1/2} p^b{}_c R^{-1 j}{}_i{}^c{}_a \epsilon_{bk} \epsilon^{al} \\
&= m^{-j}{}_i \triangleright \epsilon_{bk} p^b{}_c \epsilon^{cl} = S^{-1}(m^{+i}{}_j{}^*) \triangleright p^k{}_l{}^*, \\
(l^{+i}{}_j \triangleright p^k{}_l)^* &= (p^a{}_l \lambda^{-1/2} R^{-1k}{}_a{}^i{}_j)^* = \lambda^{-1/2} R^{-1j}{}_i{}^a{}_k \epsilon_{ba} p^b{}_c \epsilon^{cl} \\
&= \lambda^{1/2} \widetilde{R^{-1}}{}^j{}_i{}^a{}_b \epsilon_{ak} p^b{}_c \epsilon^{cl} \\
&= S^{-2} l^{-j}{}_i \triangleright \epsilon_{bk} p^b{}_c \epsilon^{cl} = S^{-1}(l^{+i}{}_j{}^*) \triangleright p^k{}_l{}^*
\end{aligned}$$

using for m^+ Eqs. (28) for our invariant tensor ϵ, and a variant of them (proven in the same way) for l^+. Here $\widetilde{R^{-1}}$ is the 'second inverse' of R^{-1} and governs the action of $S^{-2} l^-$ deduced from (33). Similarly for the m^-, l^- case. We then deduce that we have a quasi-$*$ Hopf algebra from Corollary 2.7. □

We can put general Hecke R-matrices into the above constructions. For the standard su_2 R-matrix we have for the braided linear space the rectangular quantum matrices $\bar{M}_q(2)$ which are isomorphic, in this particular case, to usual $M_q(2)$. In this way we have compatibility with a previous proposal for a suitable algebra for four-dimensional Euclidean space in [30]. Here $\bar{M}_q(2){>\!\!\!\triangleleft} U_q(su_2)\widetilde{\otimes} U_q(su_2)$ is a 'spinorial' version of the q-deformed Euclidean group of motions.

3.2. Spinorial q-Minkowski–Poincaré enveloping algebra

In this section we consider another braided linear space in matrix form $\mathbf{p} = \{p^{i_0}{}_{i_1}\}$, namely the braided matrices $B(R)$ introduced by the author as a multiplicative braided group in [3]. The additive braided group structure is due to Meyer [26] and requires that R is q-Hecke, which we assume. The relations and additive braid statistics are

$$R_{21}\mathbf{p}_1 R\mathbf{p}_2 = \mathbf{p}_2 R_{21}\mathbf{p}_1 R, \qquad R^{-1}\mathbf{p}'_1 R\mathbf{p}_2 = \mathbf{p}_2 R_{21}\mathbf{p}'_1 R \tag{37}$$

and we take braided coproduct $\underline{\Delta}\mathbf{p} = \mathbf{p}\otimes 1 + 1\otimes\mathbf{p}$. As before, we can also write $p^{i_0}{}_{i_1} = p_I$ as a braided covector space $V\check{}(\mathbf{R}', \mathbf{R})$ for suitable $\mathbf{R}', \mathbf{R}$ given in [3], [26] respectively. The equivalence between the notations is standard after the paper [20]. The algebraic relations in (37) are of interest in other contexts too [32,52], as explained in [20].

One can then follow the first part of Section 3 with the matrices $\mathbf{R}', \mathbf{R}$ and quantum group generators $l^{\pm I}{}_J$ using the setting there. This approach to the q-Lorentz group is covered in Meyer's paper [26]. Alternatively, which we do, we can follow the 'spinorial' point of view and take for our background quantum group symmetry the quantum group $H \bowtie H$ obtained by twisting the quasitriangular Hopf algebra $H \otimes H$ in the preceding section by the quantum 2-cocycle $\chi = \mathcal{R}_{23}^{-1}$ as an element of $(H \otimes H)^{\otimes 2}$. That is, χ is $\mathcal{R}^{-1}$ but with its first component living in the copy of H with generators $\mathbf{m}^{\pm}$ and its second component living in the copy of H with generators $\mathbf{l}^{\pm}$. The coproduct, antipode and quasitriangular structure are read off from (12). Note that the use of this twisted Hopf algebra to describe the Lorentz quantum group is due to the author in [9, Section 4], where we pointed out for the first time the isomorphism of previous proposals for the Lorentz quantum group function algebra in [29,53] with the dual of the 'twisted square' in [52]. It has subsequently been reiterated by other authors. The realisation of this quantum group in terms of the vectorial picture in the first part of Section 3 is cf. [26]

$$\begin{aligned} l^{+I}{}_J &= (\mathbf{l}^{-}\mathbf{m}^{+})^{i_1}{}_{j_1}((S_0^{-1}\mathbf{m}^{+})(S_0^{-1}\mathbf{l}^{+}))^{j_0}{}_{i_0}, \\ l^{-I}{}_J &= (\mathbf{l}^{-}\mathbf{m}^{-})^{i_1}{}_{j_1}((S_0^{-1}\mathbf{m}^{+})(S_0^{-1}\mathbf{l}^{-}))^{j_0}{}_{i_0}, \end{aligned} \tag{38}$$

where S_0 is the usual 'matrix inverse' antipode of H. The two copies of H no longer appear as sub-Hopf algebras due to the twisting. It is still the case that $B(R)$ lives in the category of representations of a dilatonic extension of $H \bowtie H$, and we bosonise with respect to this. The required action on the space–time co-ordinates $B(R)$, which we use now on the lowered momentum generators $p_I = p^{i_0}{}_{i_1}$, has already been given in [28]: by the twisting theory developed there, we use exactly the same formula (33) on the generators, but extending now to our new algebras. We add the dilaton ξ as before, with λ the quantum group normalisation constant of $\mathbf{R}$, which is again the square of that of R.

Proposition 3.5. *The inhomogeneous quantum group $B(R){>\!\!\!\triangleleft}(H\widetilde{\bowtie} H)$ constructed from* (9) *has cross relations, coproduct and antipode*

$$
\begin{aligned}
&R_{21}p_1Rp_2 = p_2R_{21}p_1R,\\
&l_1^+p_2l_2^- = \lambda^{-1/2}R_{21}^{-1}p_2l_2^-l_1^+,\\
&l_1^-p_2l_2^- = \lambda^{1/2}Rp_2l_2^-l_1^-,\\
&m_1^+p_2l_2^- = p_2l_2^-\lambda^{1/2}R_{21}m_1^+,\\
&m_1^-p_2l_2^- = p_2l_2^-\lambda^{-1/2}R^{-1}m_1^-,\\
&\lambda^\xi p = \lambda^{-1}p\lambda^\xi,\\
&\Delta p = p\otimes 1 + \lambda^\xi l^-m^+(\ )(S_0m^-)(S_0l^-)\otimes p, \qquad \epsilon p = 0,\\
&Sp = -\lambda^{-\xi}S_0(m^+l^-(\ )(S_0l^-)(S_0m^-))p,
\end{aligned}
\tag{39}
$$

where ( ) is a space for the matrix entries of p *to be inserted, and* S_0 *is the usual matrix antipode in either copy of* H. *The second 'conjugate' coproduct and antipode from Proposition* 2.3 *are*

$$
\begin{aligned}
&\bar{\Delta}p = p\otimes 1 + \lambda^{-\xi}l^+m^+(\ )(S_0m^+)(S_0l^-)\otimes p,\\
&\bar{S}p = -\lambda^\xi S_0(m^+l^+(\ )(S_0l^-)(S_0m^+))p.
\end{aligned}
\tag{40}
$$

Proof. We compute again from (9). This time we evaluate the coaction $p^i{}_j \to p^a{}_b(Ss^i{}_a)t^b{}_j$ which underlies (33), where s, t are dual to $l^\pm, m^\pm$, against the twisted quasitriangular structure $\chi_{21}\mathcal{R}_{H\otimes H}\chi^{-1}$, using (24). The computation follows the same line as in Proposition 3.3 except that we use the matrix coproduct of s, t in the duality pairing (2) to evaluate products of $\mathcal{R}$. □

This was announced in [11]. It is dual via Lemma 2.1 to the spinorial Poincaré function algebra computed in [27]. We are now in a position to say rather more about its structure.

Proposition 3.6. *The quantum group* $B(R)\rtimes(\widetilde{H\bowtie H})$ *is the twisting of the quantum group* $\bar{A}(R)\rtimes(\widetilde{H\otimes H})$ *from Proposition* 3.3, *by the quantum* 2*-cocycle* $\chi\in(H\otimes H)^{\otimes 2}$ *viewed in the latter quantum group.*

Proof. We just apply Proposition 2.9. That the algebras are indeed isomorphic is quite easy to see explicitly: we identify p in Proposition 3.3 with pl^- in Proposition 3.5. That the coalgebras are then related by twisting requires rather more work to verify directly. □

Thus the two systems based on $\bar{A}(R)$ and $B(R)$ are algebraically 'gauge equivalent' [16] in the sense of twisting of quantum and braided groups, so that which one chosen to work with is primarily a matter of convenience like a 'choice of co-ordinates'. The proposition extends this 'quantum Wick rotation' from [28] to the level of the associated 'Poincaré quantum groups' in the interpretation there. Next we consider the $*$-structure. We suppose that R is of real type I and λ real. A natural $*$ structure was introduced in [25], namely the Hermitian one. For $H\bowtie H$ we take the dual of the $*$-structure on the quantum group $A\bowtie A$ introduced in [9] in our abstract approach to the q-Lorentz group function algebra. On matrix generators the latter is $s^i{}_j{}^* = St^j{}_i$ as studied in [29].

Proposition 3.7. *Under the reality assumption on R, the inhomogeneous quantum group* $B(R){>\!\!\!\triangleleft}(H\widetilde{\bowtie} H)$ *becomes a quasi-$*$ Hopf algebra with*

$$l^{\pm i}{}_j{}^* = Sm^{\mp j}{}_i, \quad m^{\pm i}{}_j{}^* = Sl^{\mp j}{}_i, \quad p^i{}_j{}^* = p^j{}_i, \quad \xi^* = -\xi. \tag{41}$$

Proof. The antipode on $H\bowtie H$ is the twisted form $S(h\otimes g) = U(S_0h\otimes S_0g)U^{-1}$ from (12) where $U = \mathcal{R}_{21}$, and the $*$-structure likewise has the twisted form $(h\otimes g)^* = U(g^*\otimes h^*)U^{-1}$ (this is obvious from [9] as the dual of the $*$-structure on $A \bowtie A$ there). We take on $H\otimes H$ the 'flipped' $*$-structure $(h\otimes g)^* = g^*\otimes h^*$, with respect to which our 2-cocycle χ is of real-type in the sense needed for (13).We also have $S^{-1}_{H\otimes H}U = U$ for our particular 2-cocycle. So $l^{\pm i}{}_j{}^* = S_0 m^{\mp j}{}_i$ in $H\otimes H$ twists to $H\bowtie H$ by the same conjugation factor as for the antipode S, giving the form stated. We also know from Proposition 2.10 that the twisted $\mathcal{R}$ will be real-quasitriangular since it clearly is so on $H\otimes H$ before twisting. We check finally that action (33) is unitary in the sense (19). Thus

$$\begin{aligned}(m^{+i}{}_j \triangleright p^k{}_l)^* &= (p^k{}_a\lambda^{1/2}R^a{}_l{}^i{}_j)^* = \lambda^{1/2}R^j{}_i{}^l{}_a p^a{}_k\\ &= S^{-1}(Sl^{-j}{}_i)\triangleright p^l{}_k = S^{-1}(m^{+i}{}_j{}^*)\triangleright p^k{}_l{}^*\end{aligned}$$

as required. Similarly for the action of $\boldsymbol{m}^-$ and $\boldsymbol{l}^\pm$. We then conclude that we have a quasi-$*$ Hopf algebra from Corollary 2.7. □

Proposition 3.7 confirms that the present system based on $B(R)$ differs, however, by more than just a 'change of co-ordinates' from the system based on $\bar{A}(R)$ from Section 3.1, because it has quite a different $*$-structure: even if we refer both systems to the same algebra by untwisting the $*$-structure in Proposition 3.7, we will not obtain the previous quasi-$*$ Hopf algebra in Proposition 3.4. Indeed, it is clear from [9] and from the above proof that $B(R){>\!\!\!\triangleleft}(H\widetilde{\bowtie} H)$ is the twisting via Proposition 2.10 of $\bar{A}(R){>\!\!\!\triangleleft}(H\widetilde{\otimes} H)$ with the 'flip' $*$-structure on $H\otimes H$ and (since $(S^{-1}_{H\otimes H}U)S^{-2}_{H\otimes H}U^{-1} = 1$) the same Hermitian $\boldsymbol{p}$, in contrast to Section 3.1 where we had essentially a tensor product $*$-structure on $H\otimes H$ and a 'unitary' type $*$-structure on $\boldsymbol{p}$. Let us note also that while it may be useful to untwist in order to make such comparisons, there are good reasons too to work with the $B(R)$ 'co-ordinates' most of the time, such as its multiplicative braided group structure [3] and the remarkable identification of that with the braided universal enveloping algebra of a braided Lie algebra $\mathcal{L}(R)$ [54].

For the standard su_2 R-matrix we obtain the braided marices $BM_q(2)$ in [3], isomorphic to an algebra proposed as q-Minkowski space in [30] from consideration of the tensor product of two quantum plane. Then $BM_q(2){>\!\!\!\triangleleft}(H\widetilde{\bowtie} H)$ is a 'spinorial' version of the q-deformed Poincaré enveloping algebra in four-dimensional q-Minkowski space. Such an algebra has been studied in [31] via explicit generators and relations, i.e., the R-matrix form above and the results about it are new. The braided matrices $BM_q(2)$ are also isomorphic to a degenerate form of the Sklyanin algebra [20] and to the braided enveloping algebra of the braided-Lie algebra $\underline{gl}_{2,q}$ [55].

4. Unitary representations of quasi-$*$ Hopf algebras

In this section we provide some basic lemmas about quasi-$*$ Hopf algebras and their representations. We examine in detail the notion of tensor products of unitary representations. This leads to a general construction for sesquilinear forms or 'inner products' such that the fundamental and conjugate fundamental representations of our inhomogeneous quantum groups are mutually adjoint and hence unitary in our sense. This underpins our remarks about the differential representation of q-Poincaré quantum groups in the next section.

Lemma 4.1. *If $(H, \mathcal{R}, *)$ is a quasi-$*$ Hopf algebra in the sense of Definition* 2.6 *then:*

(i) $(\epsilon \otimes \mathrm{id})\mathcal{R} = 1 = (\mathrm{id} \otimes \epsilon)\mathcal{R}$, $(S \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}^{-1}$ *and* $(S \otimes S)(\mathcal{R}) = \mathcal{R}$.

(ii) *$\mathcal{R}$ obeys the QYBE in $H \otimes H \otimes H$.*

(iii) *$\mathcal{R}$ is a* 2*-cocycle for H (or equivalently $\mathcal{R}^{-1}$ is a* 2*-cocycle for H^{cop}).*

(iv) $* \circ S \circ * = \mathrm{u}^{-1}(S)v$ *where* $\mathrm{u}^{-1} = \mathcal{R}^{(2)}S^2\mathcal{R}^{(1)}$ *is invertible and* $\Delta \mathrm{u}^{-1} = (\mathrm{u}^{-1} \otimes \mathrm{u}^{-1})\mathcal{R}_{21}\mathcal{R}$.

Proof. These facts are analogous to similar facts for quasitriangular Hopf algebras [1] but require a little more work, except for (i), for which the proof is unchanged. For (ii) we compute

$$\begin{aligned}\mathcal{R}_{13}\mathcal{R}_{12} &= (\mathrm{id} \otimes \Delta)\mathcal{R} = (* \otimes \Delta \circ *)\mathcal{R}_{21}\\ &= (\mathcal{R}_{23}^{-1}((\mathrm{id} \otimes \tau \circ \Delta)\mathcal{R}_{21})\mathcal{R}_{23})^{*\otimes*\otimes*}\\ &= (\mathcal{R}_{23}^{-1}\mathcal{R}_{31}\mathcal{R}_{21}\mathcal{R}_{23})^{*\otimes*\otimes*} = \mathcal{R}_{32}\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{32}^{-1}.\end{aligned}$$

which is the QYBE in $H \otimes H \otimes H$ after suitable renumbering. From (ii) we deduce (iii) at once in view of the existing assumptions for Δ on $\mathcal{R}$. Part (iv) then follows from part (iii) and the theory of twisting of Hopf algebras, cf. [16], at least if S is invertible. We view $\mathcal{R}^{-1}$ as a cocycle for H^{cop} and deduce that the twisted coproduct $\bar{\Delta} = \mathcal{R}^{-1}(\tau \circ \Delta(\ ))\mathcal{R}$ has an antipode $\bar{S} = U(S^{-1}\ )U^{-1}$ where $U = \mathcal{R}^{-(1)}S^{-1}\mathcal{R}^{-(2)} = (S^2\mathcal{R}^{(1)})\mathcal{R}^{(2)}$ using part (i), and (from the twisting theory) this is invertible. We denote $U^{-1} = v$, say. But clearly $* \circ S^{-1} \circ *$ is also the antipode for $\bar{\Delta} = (* \otimes *) \circ \Delta \circ *$, hence by uniqueness of the antipode we deduce $* \circ S^{-1} \circ * = v^{-1}(S^{-1}\ )v$. This inverts to the form stated, where $\mathrm{u} = Sv$. Finally, we combine (21) from [41] with (i) and the reality condition $\mathcal{R}^{*\otimes*} = \mathcal{R}_{21}$ to deduce $\tau \circ \Delta v = \mathcal{R}_{21}^{-1}(v \otimes v)\mathcal{R}^{-1} = (v \otimes v)\mathcal{R}_{21}^{-1}\mathcal{R}^{-1}$. The same form for u follows by applying S. □

These are some of the most basic features. We denote H with its second 'conjugate' Hopf algebra structure by $\bar{H}$ (it is also a quasi-$*$ Hopf algebra, with a different 2-cocycle). It is clear that a quasi-$*$ Hopf algebra is quasitriangular *iff* it is a usual Hopf $*$-algebra (in which case it is real-quasitriangular), which is *iff* $H = \bar{H}$, Unlike usual Hopf $*$-algebras, we do not generally have $(S \circ *)^2 = \mathrm{id}$. This map $(S \circ *)^2 = S \circ \bar{S}^{-1}$ remains, however, an interesting algebra automorphism and at least in some contexts it is natural to ask that it be inner (e.g., if one wants to build a Tannakian category of representations along the lines for algebraic groups in [56]). Part (iv) of Lemma 4.1 in the form

$$(S\circ *)^2 = v(S^2(\ ))v^{-1} \tag{42}$$

tells us that is inner *iff* the automorphism S^2 is inner. For our bosonisation examples we have:

Proposition 4.2.

(i) *Let H be a quasitriangular Hopf algebra and $\alpha \in H$ invertible and such that $\Delta\alpha = (\alpha\otimes\alpha)(\mathcal{R}_{21}\mathcal{R})^{-1}$. Then α induces an automorphism $\theta_\alpha : B \to B$, $\theta_\alpha(b) = \alpha\triangleright \underline{S}^2(b)$ of any braided group B in the category of H-modules. Here $\underline{S}$ denotes its braided antipode and is assumed invertible.*

(ii) *The square of the antipode of the bosonisation Hopf algebras $B{>\!\!\!\triangleleft} H$ is*

$$S^2(b) = \theta_{\mathrm{u}}(b), \qquad S^2(h) = \mathrm{u}h\mathrm{u}^{-1}$$

for all $b\in B, h\in H$. Here $\mathrm{u} = (S\mathcal{R}^{(2)})\mathcal{R}^{(1)}$ as in [38].

Proof. That θ_α is always a braided group automorphism is clear from (7): the Ψ^2 from the action of $\underline{S}^2$ is cancelled by the $(\mathcal{R}_{21}\mathcal{R})^{-1}$ in the coproduct of α, which determines its action on tensor products. This part is an elementary fact about braided groups. In our bosonisation Hopf algebras we compute from (9) that

$$\begin{aligned} S^2(b) &= S((\mathrm{u}\mathcal{R}^{(1)}\triangleright \underline{S}b)S\mathcal{R}^{(2)}) = (S^2\mathcal{R}^{(2)})S(\mathrm{u}\mathcal{R}^{(1)}\triangleright \underline{S}b)\\ &= R^{(2)}(\mathrm{u}\mathcal{R}'^{(1)}\mathcal{R}^{(1)}\mathrm{u}\triangleright \underline{S}^2b)S\mathcal{R}'^{(2)} = \mathcal{R}^{(2)}\mathrm{u}\mathcal{R}^{(1)}\mathrm{u}\triangleright \underline{S}^2 b = \theta_{\mathrm{u}}(b), \end{aligned}$$

where we use the definition of the antipode in $B{>\!\!\!\triangleleft} H$, the standard fact that $\mathrm{u}(\ )u^{-1} = S^2$ in H and then the relations in (9). □

Let us note also that many properites of quasi-$*$ Hopf algebras depend only on the feature that $(*\otimes *)\circ\Delta\circ *$ is twisting equivalent to $\tau\circ\Delta$. That is, we can demand only the 2-cocycle property (iii) in Lemma 4.1 in place of the more restrictive axioms for $(\Delta\otimes\mathrm{id})\mathcal{R}$ and $(\mathrm{id}\otimes\Delta)\mathcal{R}$ in Definition 2.6. It is natural to call this a *cocycle-$*$* Hopf algebra. Other variants are possible as well, subject only to the existence of natural examples.

Next we consider what should be the right concept of 'unitary' representation for a quasi-$*$ Hopf algebra. The minimum definition, which is familiar for groups and usual $*$-quantum groups, is a vector space V on which the Hopf algebra is represented, and a sesquilinear form $(\ ,\ )_V : V\otimes V \to \mathbb{C}$ (antilinear in its first input) such that

$$(h^*\triangleright v, v')_V = (v, h\triangleright v')_V \tag{43}$$

for all $v, v' \in V$. We do not insist for the moment on conjugation-symmetry, non-degeneracy and positivity of the sesquilinear form. While this definition looks innocent enough, the new feature of quasi-$*$ Hopf algebras (in contrast to groups and usual $*$-quantum groups) is that such a definition is not naturally closed under tensor product. That is, there appears to be no general way to combine $(\ ,\ )_V$ and $(\ ,\ )_W$ to define a new sesquilinear form $(\ ,\ )_{V\otimes W}$ such that the tensor product representation $V \otimes W$ obeys the same condition (43). The problem does not show up for any one unitary representation but only when we try to define consistently the category of all unitaries. We formulate now a more correct notion which *is* closed under tensor product and then explain why it cannot be restricted to unitaries for fundamental reasons.

Definition 4.3. A *mutually adjoint pair* of representations of a quasi-$*$ Hopf algebra H is a vector space V, a sesquilinear form $(\ ,\ )_V : V \otimes V \to \mathbb{C}$ and *two actions* $\triangleright, \bar{\triangleright}$ of the Hopf algebra, such that

$$(h^* \bar{\triangleright} v, v')_V = (v, h \triangleright v')_V$$

for all $v, v' \in V$ and $h \in H$. A morphism between mutually adjoint pairs is a pair of intertwiners $\phi, \psi : V \to W$, one for the $\bar{\triangleright}$ representations and one for the $\triangleright$ representations, such that $(\phi(v), \psi(v'))_W = (v, v')_V$. A *quasiunitary* representation is an adjoint pair for which $\bar{\triangleright}$ and $\triangleright$ are isomorphic representations.

We recover the previous notion (43) of unitarity as the diagonal case where $\bar{\triangleright} = \triangleright$. In fact, since we do not demand that the sesquilinear form is 'conjugate symmetric', a quasiunitary representation also leads to a unitary one by absorbing any isomorphism between $\bar{\triangleright}, \triangleright$ into the sesquilinear form. The more general setting of adjoint pairs has a natural tensor product. We give two (equivalent) descriptions of it.

Proposition 4.4. *Let H be quasi-$*$ Hopf algebra (or more generally, a cocycle-$*$ Hopf algebra). Then two mutually adjoint representations $V,(\ ,\ )_V$ and $W,(\ ,\ )_W$ have a tensor product*

$$V \otimes W, \qquad (v \otimes w, v' \otimes w')_{V\otimes W} = (\mathcal{R}^{-(1)} \bar{\triangleright} v, v')_V (\mathcal{R}^{-(2)} \bar{\triangleright} w, w')_W$$

for all $v, v' \in V$ and $w, w' \in W$, where the action $\bar{\triangleright}$ of H in the first input extends to tensor products with the opposite coproduct. That is, we regard $\bar{\triangleright}$ as an $H^{\rm cop}$-module and $\triangleright$ as an H-module. The category of adjoint pair representations is monoidal.

Proof. If V, W with their sesquilinear forms are two mutually adjoint pairs in the sense of Definition 4.2 then

$$\begin{aligned}
&(h^* \bar{\triangleright} (v \otimes w), v' \otimes w')_{V\otimes W} \\
&\quad = (h^*{}_{(2)} \bar{\triangleright} v \otimes h^*{}_{(1)} \bar{\triangleright} w, v' \otimes w')_{V\otimes W} \\
&\quad = ((\mathcal{R}^{-(2)} h_{(1)} \mathcal{R}^{(2)})^* \bar{\triangleright} v \otimes (\mathcal{R}^{-(1)} h_{(2)} \mathcal{R}^{(1)})^* \bar{\triangleright} w, v' \otimes w')_{V\otimes W} \\
&\quad = (h_{(1)}{}^* \mathcal{R}^{-(2)*} \bar{\triangleright} v, v')_V (h_{(2)}{}^* \mathcal{R}^{-(1)*} \bar{\triangleright} w, w')_W \\
&\quad = (\mathcal{R}^{-(1)} \bar{\triangleright} v, h_{(1)} \triangleright v')_V (\mathcal{R}^{-(2)} \bar{\triangleright} w, h_{(2)} \triangleright w')_W \\
&\quad = (v \otimes w, h \triangleright (v \otimes w))_{V\otimes W}
\end{aligned}$$

as required, where we used the definition of $(\ ,\ )_{V\otimes W}$, the action of H in its first input using the opposite coproduct, the reality assumption on $\mathcal{R}$ in Definition 2.6, the assumption that V, W are adjoint pairs, the usual action of H in its second input and the definition of $(\ ,\ )_{V\otimes W}$ again. Hence the tensor product is also a mutually adjoint pair. We define the tensor product of two morphisms to be their usual tensor product as linear maps. This correctly connects the corresponding tensor product sesquilinear forms because each morphism (as an intertwiner) commutes with the action of $\mathcal{R}^{-1}$. This makes the tensor product of adjoint pairs of representations a functor from two copies to one copy of the category.

Moreover, this construction is associative by the usual vector space associativity. Thus

$$
\begin{aligned}
&((v\otimes w)\otimes z, (v'\otimes w')\otimes z')_{(V\otimes W)\otimes Z}\\
&\quad= (\mathcal{R}^{-(1)}\bar{\triangleright}(v\otimes w), v'\otimes w')_{V\otimes W}(\mathcal{R}^{-(2)}\bar{\triangleright}z, z')_Z\\
&\quad= (\mathcal{R}'^{-(1)}\mathcal{R}^{-(1)}{}_{(2)}\bar{\triangleright}v, v')_V(\mathcal{R}'^{-(2)}\mathcal{R}^{-(1)}{}_{(1)}\bar{\triangleright}w, w')_W(\mathcal{R}^{-(2)}\bar{\triangleright}z, z')_Z\\
&\quad= (\mathcal{R}^{-(1)}\bar{\triangleright}v, v')_V(\mathcal{R}'^{-(1)}\mathcal{R}^{-(2)}{}_{(2)}\bar{\triangleright}w, w')_W(\mathcal{R}'^{-(2)}\mathcal{R}^{-(2)}{}_{(1)}\bar{\triangleright}z, z')_Z\\
&\quad= (\mathcal{R}^{-(1)}\bar{\triangleright}v, v')_V(\mathcal{R}^{-(2)}\bar{\triangleright}(w\otimes z), w'\otimes z')_{W\otimes Z}\\
&\quad= (v\otimes(w\otimes z), v'\otimes(w'\otimes z'))_{V\otimes(W\otimes Z)}.
\end{aligned}
$$

This makes the category of adjoint pairs of representations monoidal in the sense of [57]. □

So even if $\bar{\triangleright} = \triangleright$ for some representations, as we take tensor products of them the composite $\bar{\triangleright}, \triangleright$ will begin to diverge when our coproduct is truly non-cocommutative. They need not even remain isomorphic. Clearly, one does not see this interesting phenomenon for groups or enveloping algebras. For a quasi-$*$ Hopf algebra the opposite coproduct is nevertheless twisting equivalent to the conjugate coproduct $\bar{\Delta}$, and since twisting does not change the category of representations up to equivalence we can work equivalently with the first input of $(\ ,\ )_V$ living in the category of representations of $\bar{H}$ instead. Hence we can 'neutralise' the above appearance of the cocycle in the tensor product of our sesquilinear forms provided we put it into a more complicated form for the tensor product of $\bar{\triangleright}$.

Proposition 4.5. *Let H be a quasi-$*$ Hopf algebra (or more generally, a cocycle-$*$ Hopf algebra). Then two adjoint pairs of representations $V, (\ ,\ )_V$ and $W(\ ,\ )_W$ have tensor product $V\otimes W$, $(v\otimes w, v'\otimes w')_{V\otimes W} = (v, v')_V(w, w')_W$ where the action in the first input extends to tensor products using the conjugate coproduct of H. That is, we regard $\bar{\triangleright}$ as an $\bar{H}$-module and $\triangleright$ as an H-module.*

Proof. This is entirely equivalent to Proposition 4.4, and the proof is similar. The extension of $\bar{\triangleright}$ is $h\bar{\triangleright}(v\otimes w) = \mathcal{R}^{-(1)}h_{(2)}\mathcal{R}^{(2)}\bar{\triangleright}v\otimes\mathcal{R}^{-(2)}h_{(1)}\mathcal{R}^{(2)}\triangleright w$ in terms of the coproduct of H. The tensor product representations $V\otimes W$ are not the same as before but equivalent via the morphism $\mathcal{R}^{-1}\bar{\triangleright}, \mathcal{R}^{-1}\triangleright$ mapping the adjoint pair on $V\otimes W$ as defined in Proposition 4.3 over to the adjoint pair on $V\otimes W$ as defined presently. This provides a monoidal functor between the two categories. □

Both these tensor products of mutually adjoint pairs can be useful. To be concrete we focus now on the latter formulation since it allows us to keep the usual tensor product of our sesquilinear forms. It also makes clear that our problem of $\bar{\triangleright}$ and $\triangleright$ diverging would not arise if H were a Hopf $*$-algebra in the usual sense. We consider now how to construct adjoint pairs of representations from actions on $*$-algebras. If they are also isomorphic then we will have a unitary representation after adjusting the sesquilinear form. We begin with an elementary lemma which is natural but not *quite* what we need for our examples from bosonisation. After that, we will modify it to accommodate the case of interest.

Lemma 4.6. *Let a $*$-algebra C be acted upon by a quasi-$*$ Hopf algebra H via an action $\triangleright$ making it an H-module algebra. Let $\bar{\triangleright}$ be the conjugate representation making C a $\bar{H}$-module algebra such that*

$$(h\bar{\triangleright}c)^* = (\bar{S}h)^* \triangleright c^*, \qquad (h \triangleright c)^* = (Sh)^*\bar{\triangleright}c^*$$

for all $h \in H$ and $b, c \in C$. If $\phi : C \to \mathbb{C}$ is a $\triangleright$-invariant linear functional then the sesquilinear form

$$(b, c)_\phi = \phi(b^*c)$$

makes $\bar{\triangleright}, \triangleright$ a mutually adjoint pair in the sense of Definition 4.3. *Moreover, $\bar{\phi} = \overline{\phi((\ )^*)}$ is $\bar{\triangleright}$ invariant and $\overline{(b, c)_\phi} = (c, b)_{\bar{\phi}}$.*

Proof. Firstly, $\triangleright$ and either one of the displayed conditions stated determine $\bar{\triangleright}$ (we give both forms to maintain the symmetry; they are equivalent). Then $hg\bar{\triangleright}c = ((S^{-1}(h^*))(S^{-1}(g^*))\triangleright c^*)^* = (S^{-1}(h^*) \triangleright (g\bar{\triangleright}c)^*)^* = h\bar{\triangleright}(g\bar{\triangleright}c)$ so we have an action, and

$$\begin{aligned} h\bar{\triangleright}(bc) &= (S^{-1}(h^*) \triangleright (c^*b^*))^* = (S^{-1}({h^*}_{(1)}) \triangleright b)^*(S^{-1}({h^*}_{(2)}) \triangleright c^*)^* \\ &= (h_{(\bar{1})}\bar{\triangleright}b)(h_{(\bar{2})}\bar{\triangleright}c) \end{aligned}$$

so we have covariance with respect to $\bar{\Delta}h = h_{(\bar{1})} \otimes h_{(\bar{2})}$. Also, given ϕ it is clear that H-invariance means

$$\begin{aligned} \phi((S^{-1}h \triangleright b)c) &= \phi(h_{(2)} \triangleright ((S^{-1}h_{(1)} \triangleright b)c)) \\ &= \phi((h_{(2)} \triangleright (S^{-1}h_{(1)} \triangleright b))(h_{(3)} \triangleright c)) = \phi(b(h \triangleright c)) \end{aligned} \tag{44}$$

for all $h \in H$ and $b, c \in C$. We will later need to consider this equation also with ϕ only partially invariant. So $(h^*\bar{\triangleright}b, c) = \phi((h^*\bar{\triangleright}b)^*c) = \phi((S^{-1}h \triangleright b^*)c) = \phi(b^*(h \triangleright c)) = (b, h \triangleright c)$ as required, using our definition of $\bar{\triangleright}$ as conjugate to $\triangleright$. The last line of the lemma maintains the symmetry between $\bar{\triangleright}, \triangleright$ and follows at once from their mutually conjugate relationship as stated. □

In the setting of actions on $*$-algebras, our consideration of pairs of actions $\bar{\triangleright}, \triangleright$ is the same as considering either one (since one determines the other). If, on the other hand, $\bar{\triangleright}, \triangleright$ are given to us from some other source then the displayed condition in Lemma 4.6 becomes a non-trivial constraint that the two actions are mutually conjugate. The lemma generalises standard considerations for unitary actions of Hopf $*$-algebras to our setting of pairs of actions of quasi-$*$ Hopf algebras. If $\phi = \bar{\phi}$ then the resulting sesquilinear form is conjugate symmetric, which is again the usual case. One can then proceed to construct a Hilbert space from these data in the standard way.

Unfortunately, our basic examples such as the fundamental and conjugate fundamental representation of the quasi-$*$ Hopf algebras obtained by bosonisation, do not necessarily fit into this standard setting and we have to modify it. The variant we need is the following. It generally destroys, however, the conjugation-symmetry of the sesquilinear form and hence forces us to modify this axiom of Hilbert space theory.

Lemma 4.7. *Let $(C, \star)$ be a $*$-algebra acted upon by a quasi-$*$ Hopf algebra H by actions $\triangleright, \bar{\triangleright}$ as in Lemma 4.6 but now mutually conjugate in the sense $(h\bar{\triangleright}c)^\star = S(h^*) \triangleright c^\star$ for all $c \in C$ and $h \in H$. Let $\theta : C \to C$ be an algebra automorphism such that $\theta(h \triangleright c) = S^{-2}h \triangleright \theta(c)$ for all h, c. Then $(b, c)_\phi^\theta = \phi((\theta(b^\star)c)$ makes $\bar{\triangleright}, \triangleright$ mutually adjoint. The conjugate construction has just the same form with*

$$\bar{\phi} = \overline{\phi((\ )^\star)}, \qquad \bar{\theta} = \star \circ \theta^{-1} \circ \star, \qquad \overline{(c, b)_\phi^\theta} = (b, c)_{\bar{\phi}\circ\bar{\theta}^{-1}}^{\bar{\theta}}.$$

Proof. This is a variant of the preceding proposition. We have $(h^*\bar{\triangleright}b, c) = \phi(\theta((h^*\bar{\triangleright}b)^\star)c)$ $= \phi(\theta(Sh \triangleright b^\star)c) = \phi((S^{-1}h \triangleright \theta(b^\star))c) = (b, h \triangleright c)$ much as before. For the conjugate we deduce $\bar{\theta}(h\bar{\triangleright}c) = (\bar{S}^{-2}h)\bar{\triangleright}\bar{\theta}(c)$ when this is defined as stated. Hence the conjugate construction has the same form. □

If we had $\bar{\theta} = \theta$ then we would be able to redefine $\theta \circ \star$ as new $*$-structure and precisely return to the setting of Lemma 4.6. This is, however, not necessarily the case for the examples coming from bosonisation of braided groups which interest us here. The origin of the problem (already noted in [8]) is that the duality pairing (8) for $*$-braided groups does not preserve the unitarity condition (19) for the action of the background quantum group generating the category. Indeed, the natural 'unitarity' of the action on the $*$-braided group C dual to B where the action obeys (19), is instead

$$(h \triangleright c)^\star = S(h^*) \triangleright c^\star \tag{45}$$

for $c \in C$ and h in the background quantum group. This is clear from invariance of ev and computation of $\overline{\mathrm{ev}\,(h \triangleright b, c)} = \overline{\mathrm{ev}\,(b, (Sh) \triangleright c)}$. When the duality pairing is degenerate then $\star$ is not fully determined by (8) but we nevertheless keep (45) as a reasonable assumption compatible with condition (19) which we assumed for the action on B.

Proposition 4.8. *Let $B{>\!\!\!\triangleleft}H$ be the quasi-$*$ Hopf algebra constructed as in Corollary 2.7, where H is a real-quasitriangular Hopf algebra acting unitarily on $*$-braided group B. Let C be a $*$-braided group dual to B on which H acts as in (45). The fundamental and conjugate fundamental representation from Corollaries 2.2 and 2.4 of $B{>\!\!\!\triangleleft}H$ on C are then mutually conjugate in the sense of Lemma 4.7:*

$$(x\bar{\triangleright}c)^\star = S(x^*) \triangleright c^\star, \qquad (x \triangleright c)^\star = \bar{S}(x^*)\bar{\triangleright}c^\star$$

for all $x \in B{>\!\!\!\triangleleft} H$, $c \in C$. Here $(\ )^\star$ is the $*$-structure on C characterised by (8) *and S, $\bar{S}$ the antipodes for $B{>\!\!\!\triangleleft} H$.*

Proof. We compute the first of these for $b \in B$ from the definition in Corollary 2.4 as

$$\begin{aligned}(b\bar{\triangleright}c)^\star &= c_{\underline{(2)}}{}^\star\,\mathrm{ev}(b^*, c_{\underline{(1)}}{}^\star) = \mathcal{R}^{(2)}\mathcal{R}^{-(2)} \triangleright c^\star{}_{\underline{(1)}}\,\mathrm{ev}\,(\underline{S}^{-1}\underline{S}(b^*), \mathcal{R}^{(1)}\mathcal{R}^{-(1)} \triangleright c^\star{}_{\underline{(2)}})\\
&= (S\mathcal{R}^{(2)})\mathcal{R}^{-(2)} \triangleright c^\star{}_{\underline{(1)}}\,\mathrm{ev}\,(\underline{S}^{-1}(\mathcal{R}^{(1)} \triangleright \underline{S}(b^*)), \mathcal{R}^{-(1)} \triangleright c^\star{}_{\underline{(2)}})\\
&= (S\mathcal{R}^{(2)}) \triangleright ((\mathcal{R}^{(1)} \triangleright \underline{S}(b^*)) \triangleright c^\star) = (((S\mathcal{R}^{(2)}{}_{(2)})\mathcal{R}^{(1)} \triangleright \underline{S}(b^*))S\mathcal{R}^{(2)}{}_{(1)}) \triangleright c^\star\\
&= (u\mathcal{R}^{(1)} \triangleright \underline{S}(b^*))S\mathcal{R}^{(2)} \triangleright c^\star = S(b^*) \triangleright c^\star,\end{aligned}$$

where the first equality is the definition of $\bar{\triangleright}$, while the second inserts $\mathcal{R}\mathcal{R}^{-1}$ and uses one of the $*$-braided group axioms (4) to move $\star$ onto c. For the third equality we replace $\mathcal{R}$ by $(S \otimes S)(\mathcal{R})$ and move $S\mathcal{R}^{(1)}$ over to the left-hand input of ev (by its H-invariance). It passes through $\underline{S}^{-1}$ also, since this is H-covariant. The fourth equality recognises the fundamental representation from Corollary 2.2. The fifth uses that this too is a representation of $B{>\!\!\!\triangleleft} H$, allowing us to move the $S\mathcal{R}^{(2)}$ to the left via its relations (9). The sixth computes this via the coproduct action for $\mathcal{R}$, which gives the element u. We then recognise the antipode S of $B{>\!\!\!\triangleleft} H$ from (9). The proof for $(b \triangleright c)^\star$ is analogous, giving this time $\bar{S}$ from Proposition 2.3. For $x \in H$ we know that its actions $\bar{\triangleright}$ and $\triangleright$ coincide with the action of H on C as a braided group, so this case is (45) characterised by the braided group duality (8) as explained above. □

Hence Lemma 4.7 is the variant of the general theory for quasi-$*$ Hopf algebras that we need for our bosonisation examples $B{>\!\!\!\triangleleft} H$ with $\triangleright$ the fundamental representation. Next we consider the functional ϕ. In the present case it is clear from the definition of $\triangleright$ in Corollary 2.2 that when the duality pairing ev is non-degenerate, a $B{>\!\!\!\triangleleft} H$-invariant complex linear map $C \to \mathbb{C}$ means nothing other that an H-covariant and right-invariant braided integration $\int$ of the braided group C. Likewise, a $\bar{\triangleright}$-invariant linear map means nothing other than an H-covariant and left-invariant integration $\int_{\mathrm{L}}$. These maps are characterised as morphisms $C \to \mathbb{C}$ by

$$\left(\int c_{\underline{(1)}}\right) c_{\underline{(2)}} = \int c, \qquad c_{\underline{(1)}} \int_{\mathrm{L}} c_{\underline{(2)}} = \int_{\mathrm{L}} c \tag{46}$$

for all $c \in C$, respectively. It is clear from (4) that for $*$-braided groups a morphism $\int$ is a right integral *iff* $\bar{\int} = \overline{\int(\ )^\star}$ is a left integral. At least for finite-dimensional and other reasonable braided groups, the left and right integrals (if they exist) are unique up to normalisation. So we could take $\int_{\mathrm{L}} = \bar{\int}$ without significant loss of generality.

Proposition 4.9. *In the setting of Proposition 4.8, let $\int$, $\int_{\mathrm{L}}$ be right-, left-invariant integrations for the braided group $\mathbb{C}$. Then the conjugate/fundamental representations $\bar{\triangleright}$, $\triangleright$ are* *mutually adjoint with respect to the sesquilinear form (,), and* $\triangleright, \bar{\triangleright}$ *with respect to the sesquilinear form* $(,)^{\mathrm{L}}$*, defined by*

$$(b,c)=\int \theta_{\mathrm{V}}(b^\star)c,\qquad (b,c)^{\mathrm{L}}=\int_{\mathrm{L}} \theta_{\mathrm{U}}^{-1}(b^\star)c$$

for all $b,c\in C$*. Moreover,* $\overline{(c,b)}=(b,c)^{\mathrm{L}}$ *when* $\int_{\mathrm{L}}=\bar{\int}\circ\theta_{\mathrm{U}}$.

Proof. One can verify directly that the first sesquilinear form makes $\bar{\triangleright}, \triangleright$ mutually adjoint. We have already done the work however, and deduce it as follows. From the mutual conjugation property in Proposition 4.8 and the assertion in Corollary 2.4 that $\bar{\triangleright}, \triangleright$ are intertwined by $\underline{S}$, we easily deduce that

$$\underline{S}^2(x\triangleright c)=((S\circ *)^{-2}x)\triangleright \underline{S}^2 c,\qquad \underline{S}^2(x\bar{\triangleright}c)=((S\circ *)^{-2}x)\bar{\triangleright}\underline{S}^2 c \tag{47}$$

for all $x\in B{>\!\!\triangleleft} H$ and $c\in C$. From this, (42), and Proposition 4.2 it is clear that

$$\theta_{\mathrm{V}}(x\triangleright c)=(S^{-2}x)\triangleright\theta_{\mathrm{V}}(c),\qquad \theta_{\mathrm{V}}(x\bar{\triangleright}c)=(S^{-2}x)\bar{\triangleright}\theta_{\mathrm{V}}(c). \tag{48}$$

Hence we have the required automorphism $\theta=\theta_v$ for Lemma 4.7. By similar considerations to those above, we deduce equally well

$$\theta_{\mathrm{U}}(x\triangleright c)=(\bar{S}^2x)\triangleright\theta_{\mathrm{U}}(c),\qquad \theta_{\mathrm{U}}(x\bar{\triangleright}c)=(\bar{S}^2x)\bar{\triangleright}\theta_{\mathrm{U}}(c) \tag{49}$$

for all $x\in B{>\!\!\triangleleft} H$ where $\theta_{\mathrm{U}}=\star\circ\theta_{\mathrm{V}}\circ\star$. The latter follows from $v^*=v, Sv=\mathrm{u}$ and (4). This gives the conjugate construction for the sesquilinear form $(,)^{\mathrm{L}}$ obeying $(h^*\triangleright b,c)^{\mathrm{L}}=(b,h\bar{\triangleright}c)^{\mathrm{L}}$. Then

$$\overline{\int \theta_{\mathrm{V}}(b^\star)c}=\bar{\int} b^\star\theta_{\mathrm{U}}(c)=\bar{\int}(v\triangleright b^\star)\underline{S}^2 c=\bar{\int}\theta_{\mathrm{U}}(\theta_{\mathrm{U}}^{-1}(b^\star)c)$$

explicitly relates the two constructions as stated. The third expression means that we can also write $\overline{(c,b)}=(\underline{S}^{-2}b,\underline{S}^2c)$ if we replace $\int$ on the right by $\bar{\int}$. □

We have $\bar{\int}\circ\theta_{\mathrm{U}}=\bar{\int}$ as well in reasonable cases where the left-invariant integration is unique up to scale. This achieves the task of making our fundamental and conjugate fundamental representations from Corollaries 2.2 and 2.4 mutually adjoint. Since we already know that they are equivalent with interwining operator $\underline{S}$, we can absorb this too into the sesquilinear form. Then

$$(b,c)^U_\phi=\int (v\triangleright \underline{S}b^\star)c \tag{50}$$

for $b,c\in C$ defines a sesquilinear form with respect to which the fundamental representation of $B{>\!\!\triangleleft} H$ in Corollary 2.2 is unitary. In our interpretation as Poincaré quantum group represented on space–time, it corresponds to building a parity operator into the L^2 inner product. With respect to such a non-local and non-symmetric sesquilinear form we would have ∂^i self-adjoint (i.e., symmetric) rather than the usual anti-self-adjoint. The sesquilinear form in Proposition 4.9 by contrast becomes the usual L^2 inner product.

As explained above, it is natural in these constructions to consider $\theta_V \circ \star$ as a possible second $*$-structure on C. It is an antilinear anti-algebra homomorphism but

$$(\theta_V \circ \star)^2 = uv \triangleright \underline{S}^4 \tag{51}$$

is not necessarily the identity, though it is in some cases. In the present setting, uv is a central element of our quasitriangular Hopf algebra. One could also build in its square root when this exists (the so-called ribbon element [37]) in order to reduce the contribution of v to θ^2. Partly in this direction, it is easy to see that if (as in the ribbon case) we have [38] a group-like element σ inplementing S^2 and such that $\sigma^* = \sigma$ then any $*$-braided group $(C, \star)$ in the category of H-modules has another $*$-structure $c^* = \sigma^{-1} \triangleright c^\star$. Moreover, this converts (45) over to the more standard unitarity condition (19). If we use this second $*$-structure then the sesquilinear form in Proposition 4.8 becomes

$$(b, c) = \int \theta_\nu(b^*)c, \tag{52}$$

where $\nu = (vu)^{1/2}$ is the ribbon element. This is a purely cosmetic change.

We conclude this abstract section with an example of the above theory which is somewhat different from the inhomogeneous quantum groups of primary interest in the present paper. Namely, we showed in [20] how to view Drinfeld's quantum double $D(U_q(g))$ as a bosonisation $BU_q(g) \rtimes U_q(g)$, where $U_q(g)$ is from [1,2] and $BU_q(\mathrm{g})$ is its associated braided group from [20]. We consider the standard Lie algebra deformations where there is an R-matrix form $\boldsymbol{l}^{\pm}, \boldsymbol{t}$ as in [32] of the quantum group and its dual G_q. The dual of $BU_q(g)$ is the matrix braided group BG_q obtained as a corresponding quotient of the braided matrices $B(R)$ [3]. We take $q^* = q$ and the standard compact real form of the quantum groups. As a $*$-algebra $BU_q(g)$ coincides with $U_q(g)$ and we take matrix generator $\boldsymbol{m} = \boldsymbol{m}^+ S\boldsymbol{m}^-$ forming a $*$-braided group. The braided coproduct is $\underline{\Delta}\boldsymbol{m} = \boldsymbol{m} \otimes \boldsymbol{m}$ and the $*$-structure is the Hermitain one. We refer to [20,25] for full details of this bosonisation.

Proposition 4.10. *The quantum double $D(U_q(g))$ in the bosonisation form $BU_q(g) \rtimes U_q(g)$ is a quasi-$*$ Hopf algebra by Corollary* 2.7. *The fundamental and conjugate fundamental representations are*

$$\boldsymbol{l}_1^+ \triangleright \boldsymbol{u}_2 = R_{21}^{-1}\boldsymbol{u}_2 R_{21}, \qquad \boldsymbol{l}_1^- \triangleright \boldsymbol{u}_2 = R\boldsymbol{u}_2 R^{-1},$$
$$\boldsymbol{m}_1 \triangleright R\boldsymbol{u}_2 = R\boldsymbol{u}_2 R_{21} R, \qquad \boldsymbol{m}_1 \bar{\triangleright} R\boldsymbol{u}_2 = R_{21}^{-1}\boldsymbol{u}_2$$

on BG_q with matrix generator $\boldsymbol{u}$. Moreover, $\theta_V \circ \star = \underline{*}$ is the standard Hermitian $*$-structure on BG_q and

$$(b, c) = \int b^{\underline{*}} \underline{\cdot} c = \int \Theta(Sb)^* c, \qquad \Theta(c) = c_{(3)}\mathcal{R}((Sc_{(1)})c_{(4)}, S^2 c_{(2)}),$$

where $\underline{\cdot}$ is the product in BG_q. The second expression computes this further in terms of $b, c \in G_q$ and its usual coproduct, antipode, $*$, dual-quasitriangular structure [28] *and Haar measure* [10] $\int$. *Here we identified $BG_q = G_q$ as coalgebras by transmutation* [35,36].

Proof. The bosonisation and its $*$-structure for real q is described in detail in [25] (as well as for q modulus 1, which we do not consider here). The braided group duality pairing between $B = BU_q(g)$ and $C = BG_q$ is also given there, as $\mathrm{ev}(b, c) = \langle Sb, c\rangle$ where the right-hand side views $b \in H = U_q(g)$ and $c \in G_q$ since they coincide as linear spaces (B has a modified coproduct and C a modified product, making them braided groups in the category of H-modules by the quantum adjoint and coadjoint action, respectively). Then the fundamental representation from Corollary 2.2 is

$$b \triangleright c = \mathcal{R}^{-(2)} \triangleright c_{(1)}\, \mathrm{ev}\,(\underline{S}^{-1} b, \mathcal{R}^{-(1)} \triangleright c_{(2)}) = \langle \mathcal{R}^{(2)}, Sc_{(1)}\rangle c_{(2)} \langle \mathcal{R}^{(1)} \triangleright b, c_{(3)}\rangle$$

by a computation similar to that for braided right-invariant vector fields in [54, Proposition 6.2]. The conjugate fundamental from Corollary 2.4 is just $b \bar{\triangleright} c = \mathrm{ev}\,(b, c_{(1)}) c_{(2)} = \langle Sb, c_{(1)}\rangle c_{(2)}$. Putting in the matrix coproduct for the generator $t \in G_q$ and a standard computation from (24) gives the form for these actions as stated. The action of the $U_q(g)$ part is the same for $\triangleright$ and $\bar{\triangleright}$ and is easily computed [3] by the evaluation of $l^{\pm}$ against the quantum adjoint coaction.

Next, the operation $\star$ on BG_q dual to that on $BU_q(g)$ in the sense of (8) comes out as

$$c^{\star} = S^{-3}(c^{*}), \quad \text{i.e.,} \quad \theta_V(c^{\star}) = \sigma^{-1} \triangleright c^{\star} = (Sc)^{*}$$

in terms of the usual $*$ and antipode S of G_q. Here $\triangleright$ is the quantum coadjoint action hence it is immediate that $\sigma^{-1}\triangleright = S^2$. With rather more work, we may use the formula in [36] for the braided antipode $\underline{S}$ of BG_q in terms of G_q to obtain $\theta_V = S^2$ as well. Since S^2 is an automorphism of the quantum group G_q it necessarily induces an automorphism of the associated BG_q. In the same way, $\theta_\nu = \mathrm{id}$. We recognise the combination $\underline{*} = (S\,)^{*}$ (which we underline to keep it distinct from the $*$ of G_q) as the Hermitian $*$-braided group structure for BG_q introduced in [25]. Hence

$$\begin{aligned}(b, c) &= \int \theta_V(b^{\star}) \underline{\cdot} c = \int b^{\underline{*}} \underline{\cdot} c \\ &= \int ((Sb)^{*})_{(2)} c_{(2)} \mathcal{R}((S((Sb)^{*})_{(1)})((Sb)^{*})_{(3)}, Sc_{(1)}) \\ &= \int (\mathcal{R}^{(1)} \triangleright (Sb)^{*}) c_{(2)} \langle \mathcal{R}^{(2)}, Sc_{(1)}\rangle,\end{aligned}$$

where the third equality expresses the product $\underline{\cdot}$ of BG_q in terms of G_q using the transmutation formula in [36]. It is useful (but not necessary) to make this conversion because the Haar weights for compact quantum groups are already known [10], in some cases quite explicitly. Since the BG_q has the same coalgebra, we use the same $\int$. It is a morphism $C \to \mathbb{C}$ because it is both left and right invariant on G_q. Invariance also gives the form for (,) stated. This, and the actions $\triangleright, \bar{\triangleright}$ make sense over $\mathbb{C}$ and do not require formal powerseries. □

The bosonisation here for the simplest case $BU_q(su_2) \rtimes U_q(su_2)$ was studied in detail in [25] so we do not repeat this here. The new feature is that we know now that its

$*$-algebra, which we developed there as q-deformed Mackey quantisation of a particle moving on the mass hyperboloid in q-Minkowski space, is a quasi-$*$ Hopf algebra. Moreover, the fundamental representation $\boldsymbol{m}\triangleright = \partial$ extends as a 'matrix braided-derivation' and was computed (in a right-hand version) for the $BSU_q(2)$ case in [54], to which we refer for details. These derivatives form braided left-invariant vector fields on the braided group and realise the matrix braided-Lie algebra $\underline{gl}_{2,q}$ [54]. From either point of view it is natural to consider functional analysis on BG_q as the algebra of 'co-ordinate functions' of the braided group. The above theory now provides the sesquilinear form via the integral which to lowest order (carrying out required the transmutation explicitly) comes out as

$$\int 1 = 1, \qquad \int a^2 = \int ab = \int ac = \int b^2 = \int bd = \int c^2 = \int cd = 0,$$
$$\int ad = 1/(1+q^2) \qquad \int d^2 = (1-q^{-2})/(1+q^2) \tag{53}$$

in terms of the generators

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and the product of $BSU_q(2)$. The integrals of the other quadratic expressions are determined by the relations of $BSU_q(2)$. Our left-invariant vector fields and their conjugates are adjoint with respect to the corresponding sesquilinear form $(b, c) = \int b^* c$, etc., where

$$\begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

is the Hermitian $*$-braided group structure of $BSU_q(2)$. This demonstrates the possibility of a braided approach to q-harmonic analysis on braided versions of compact quantum groups, to be considered elsewhere. Note that in this family of examples the sesquilinear form $(\ ,\ )$ clearly remains conjugate-symmetric, i.e., we are in the usual setting as in Lemma 4.6 after redefining $*$. We do lose, however, positive definiteness.

5. Differential representation on space–time and concluding remarks

In this section we will consider how our above results apply to the fundamental and conjugate fundamental representations of the inhomogeneous quantum groups from Section 3. We are now in a position to understand the subtle role of the $*$-structure, which is obviously crucial for the interpretation of these quantum groups as, for example, the q-Poincaré group in q-deformed geometry.

We begin with the fundamental and conjugate fundamental representations themselves on the 'space–time' braided group $V^{\vee}(R', R)$ with co-ordinates $\{x_i\}$ dual to the linear 'momentum' part of the inhomogeneous quantum group. The existence of a general fundamental representation on 'space–time' was the main result of the braided approach in [9] in the form of a 'rotation+translation' coaction of the dual quantum groups. Evaluating against this coaction (or from Corollary 2.2), we obtain at once

$$
\begin{aligned}
&\boldsymbol{l}_1^+ \triangleright \boldsymbol{x}_2 = \boldsymbol{x}_2\lambda R_{21}, \qquad \boldsymbol{l}_1^- \triangleright \boldsymbol{x}_2 = \boldsymbol{x}_2\lambda^{-1}R^{-1},\\
&\lambda^{\xi} \triangleright x_i = \lambda x_i, \qquad p^i \triangleright x_j = -\delta^i{}_j,
\end{aligned}
\tag{54}
$$

which we have used already in (25) and (33) in Section 3. This then extends to products as a module algebra. Explicitly:

Proposition 5.1. *The inhomogeneous quantum group $V(R', R)\rtimes\!\!\!\triangleleft \tilde{H}$ in the setting of Section 3 acts covariantly on the braided covector algebra $V\check{}(R', R)$ with generators $\boldsymbol{x} = \{x_i\}$ by the fundamental and conjugate fundamental representiations in Corollaries 2.2 and 2.4 as*

$$
\begin{aligned}
\lambda^{\xi} \triangleright (x_{i_1}\cdots x_{i_m}) &= \lambda^m x_{i_1}\cdots x_{i_m},\\
l^{+i}{}_j \triangleright (x_{i_1}\cdots x_{i_m}) &= \lambda^m x_{j_1}\cdots x_{j_m}[1, m+1; R]^{ij_1\cdots j_m}_{i_1\cdots i_m j},\\
l^{-i}{}_j \triangleright (x_{i_1}\cdots x_{i_m}) &= \lambda^{-m} x_{j_1}\cdots x_{j_m}[1, m+1; R_{21}^{-1}]^{ij_1\cdots j_m}_{i_1\cdots i_m j},\\
-p^i \triangleright (x_{i_1}\cdots x_{i_m}) &= x_{j_2}\cdots x_{j_m}[m; R_{21}^{-1}]^{ij_2\cdots j_m}_{i_1 i_2\cdots i_m},\\
p^i \bar{\triangleright} (x_{i_1}\cdots x_{i_m}) &= x_{j_2}\cdots x_{j_m}[m; R]^{ij_2\cdots j_m}_{i_1 i_2\cdots i_m},
\end{aligned}
$$

where $[1, m+1; R] = (PR)_{12}(PR)_{23}\cdots(PR)_{mm+1}$, and $[m; R] = [1, 2; R]+[1, 3; R]+\cdots+[1, m; R]$ is the braided integer matrix in [22]. *The action of the $\lambda^{\xi}, \boldsymbol{l}^{\pm}$ generators is the same for the two representations.*

Proof. Here $V\check{}(R', R)$ is the algebra $\boldsymbol{x}_1\boldsymbol{x}_2 = \boldsymbol{x}_2\boldsymbol{x}_1 R'$ forming a braided group [22]. Its braided group duality with the braided vector algebra used for the linear part of the inhomogeneous quantum group is $\mathrm{ev}(p^i, x_j) = \delta^i{}_j$. Since we know (by the bosonisation theory) that the fundamental representation makes $V\check{}(R', R)$ a module algebra, the action on products is then determined. The action of $\boldsymbol{l}^{\pm}$ on products is immediate from their matrix coproduct. The braided integer matrix is a sum of the corresponding matrices, so it is clear by induction that p^i acts by such matrices given the action of $\boldsymbol{l}^{\pm}$ and the form of the coproduct and conjugate coproduct. In the case of the conjugate coproduct the action of p^i is necessarily braided differentiation since the map in Corollary 2.4 is exactly its definition in [22]. Note that we have $-$ sign in the first action $\triangleright$ of p^i from the braided antipode in Corollary 2.2 but not in the conjugate action $\bar{\triangleright}$ in Corollary 2.4. □

In particular, the covariant action of p^i using the conjugate coproduct $\bar{\Delta}$ is precisely the braided differentiation $p^i = \partial^i$ as introduced for general braided linear spaces in [22]. In fact, we know this without computation because evaluation against the braided coproduct as in the asbtract definition of $\bar{\triangleright}$ in Corollary 2.4 is precisely the definition of ∂^i in [22] as an 'infinitesimal coaddition' in the braided approach. The covariant action of p^i using the original coproduct Δ is by 'conjugate' derivatives $-p^i = \bar{\partial}^i$ in which the role of R is replaced by R_{21}^{-1} when extending to products. It is an infinitesimal translation from the right and corresponds to the 'right derivatives' $\overleftarrow{\partial}$ in [8], converted over to left-acting derivatives by means of the braided antipode. This is the reason for the extra $-$ sign in the action of $\bar{p}^i$. The ∂ obey a braided-Leibniz rule with Ψ^{-1} as explained in [22], while the $\bar{\partial}$ obey a braided-Leibniz rule with Ψ as we have seen already in the proof of Corollary 2.2. The reversed matching of $\bar{\triangleright}$ with ∂, etc., is a historical accident reflecting the fact that the unbarred Δ and unbarred ∂ are each natural in their own settings.

One can also consider x_i as an operator on $V\check{}(R', R)$ by left multiplication then the corresponding braided-Leibniz rules are expressed as the commutation relations

$$\partial_1 \boldsymbol{x}_2 - \boldsymbol{x}_2 R_{21} \partial_1 = \mathrm{id}, \qquad \bar{\partial}_1 \boldsymbol{x}_2 - \boldsymbol{x}_2 R^{-1} \bar{\partial}_1 = \mathrm{id}, \tag{55}$$

cf. specific examples in [58,59,31], etc. If we assume a quantum metric and lower indices by $\partial_i = \eta_{ia}\partial^a$ and $\bar{\partial}_i = \eta_{ia}\bar{\partial}^a$ then these become

$$\partial_1 \boldsymbol{x}_2 - \lambda^{-2} \boldsymbol{x}_2 \partial_1 R_{21}^{-1} = \eta, \qquad \bar{\partial}_1 \boldsymbol{x}_2 - \lambda^2 \boldsymbol{x}_2 \bar{\partial}_1 R = \eta \tag{56}$$

using the quantum metric identities (28). These represent the lower-index momentum generators p_i for the two actions. This is how the constructive braided approach to differentials [22] recovers previous approaches [58,60] where examples of such commutation relations were postulated as an ansatz or deduced from postulated relations between differential forms within an axiomatic approach for these.

We obviously have similar formulae for the representations of the 'spinorial' forms $p^{i_0}{}_{i_1}$ in Sections 3.1 and 3.2. This is just a change of notation from the vector form to the matrix form, $\partial_I = \partial^{i_0}{}_{i_1}$ and $\bar{\partial}_I = \bar{\partial}^{i_0}{}_{i_1}$, say. Clearly

$$R\partial_2 \boldsymbol{x}_1 R - \lambda^{-2} \boldsymbol{x}_1 \partial_2 = R\eta_{21} R, \qquad \bar{\partial}_1 \boldsymbol{x}_2 - \lambda^2 R x_2 \bar{\partial}_1 R = \eta \tag{57}$$

in the $\bar{A}(R)$ case where $\partial, \bar{\partial}$ obey the $\bar{A}(R)$ relations $R_{21}\partial_1\partial_2 = \partial_2\partial_1 R$, etc., just because this is how $\boldsymbol{R}', \boldsymbol{R}$ appear in this notation (31). Likewise, we have

$$\begin{aligned} &\partial_2 R_{21} \boldsymbol{u}_1 R - \lambda^{-2} R^{-1} \boldsymbol{u}_1 R \partial_2 = \eta^{(2)} R_{21} \eta^{(1)} R, \\ &R^{-1} \bar{\partial}_1 R \boldsymbol{u}_2 - \lambda^2 \boldsymbol{u}_2 R_{21} \bar{\partial}_1 R = R^{-1} \eta^{(1)} R \eta^{(2)} \end{aligned} \tag{58}$$

in the $B(R)$ case where $\partial, \bar{\partial}$ obey the $B(R)$ relations $R_{21}\partial_1 R \partial_2 = \partial_2 R_{21} \partial_1 R$, etc., because this is how $\boldsymbol{R}', \boldsymbol{R}$ appear in this notation (37). Here $\eta = \eta^{(1)} \otimes \eta^{(2)}$ is $\eta_{IJ} = \eta^{i_0}{}_{i_1}{}^{j_0}{}_{j_1}$ as an element of $M_n \otimes M_n$, and the right-hand sides are typically multiples of it as well. We include these formulae for completeness only; they are just the standard construction (56) applied to the particular $\boldsymbol{R}', \boldsymbol{R}$ introduced in [3,26,28].

Since the braided approach derives such relations from the braided coproduct rather than imposing them axiomatically, we are now in a position to say more about the derivatives $\partial, \bar{\partial}$ than is evident from the relations alone.

Corollary 5.2. *The braided antipode $\underline{S}(\boldsymbol{x}) = -\boldsymbol{x}$ of the braided group $V\check{}(R', R)$ intertwines the actions of $\partial, -\bar{\partial}$,*

$$\underline{S}\partial^i = -\bar{\partial}^i \underline{S}.$$

Proof. The diagrammatic proof is given in the proof of Corollary 4.6 and is a result entirely as operators on the braided group $V^{\check{}}(R', R)$ in the setting of [22]. For a direct proof the key fact is that $\underline{S}$ extends to products as a braided-anti-algebra homomorphism (7), giving it an expression in terms of R-matrices [9] which intertwines the braided integer matrices in Proposition 4.1. The operator $\underline{S}$ is also covariant under the background quantum group H [9] and hence intertwines its action also. It also preserves degree, hence the action of λ^{ξ}. □

This general result contrasts with other approaches, e.g. [33,61] where it is sometimes possible to write the $\bar{\partial}$ as some non-linear function of the ∂. We have not taken this line here: in the braided approach Corollary 5.2 is more natural since the braided antipode $\underline{S}$ is inversion on the additive braided group, i.e., the 'braided parity operator' and plays an important role in numerous other constructions as well, such as the braided Fourier theory in [24,62]. In usual underformed constructions we see it merely as a minus sign, but in the braided case it extends as a non-trivial operator.

Next, we consider $*$-structures. We have seen in Section 4 the need to consider a different $*$-structure $\star$ on the 'space–time' braided group C determined by duality (8) with the linear 'momentum' part B of our inhomogeneous quantum group. We have used for the latter the standard $*$-structure which is characterised by the unitarity condition (19) with respect to the action of the background quantum group. Hence for the space–time $*$-braided group we need one which is characterised by (45), a condition which is different when $S^2 \neq \mathrm{id}$. This splitting into two $*$-structures (even when the braided groups are isomorphic via the quantum metric) is therefore a new feature of q-deformation. The duality pairing of $*$-braided groups in the present linear case was studied in [8] from where we may deduce the required $\star$ appropriate to the $*$ used for the momentum generators p^i in Section 3. For the type I case (as in Propositions 3.1 and 3.4) and the real type II case (used in Proposition 3.7) they are

$$p^{i*} = \begin{cases} \eta_{ia} p^a, & \text{real type I,} \\ \eta^{\bar{i}a}\eta_{ab} p^b, & \text{real type II,} \end{cases}$$
$$x_i{}^{\star} = \begin{cases} x_a \eta^{ia}, & \text{real type I,} \\ x_b \eta_{\bar{i}a}\eta^{ab}, & \text{real type II,} \end{cases} \tag{59}$$

where η is the quantum metric and in the type II case we also have an involution $\bar{\ }$ on its indices. This corresponds via the quantum metric to $p_i{}^* = p_{\bar{i}}$ as used in Proposition 3.7 in the form $p^{i_0}{}_{i_1}$ Hermitian. With $\star$ defined correctly, we see from Proposition 4.8 that it connects $\partial, \bar{\partial}$. In the real type I setting of Proposition 3.2, this is

$$(\partial^i f(x))^{\star} = \lambda^{\xi} S^2 l^{-a}{}_i \triangleright \bar{\partial}_a f(x)^{\star} \tag{60}$$

for all multinomials $f(x)$. We used the antipode from Proposition 3.1. A more intrinsic formulation (purely in terms of the linear braided group) is possible [8] if we use right-acting derivatives $\overleftarrow{\partial}$ in place of $\bar{\partial}$. There are corresponding formulae for the spinorial versions in Sections 3.1 and 3.2 as well.

We also needed in Section 4 the braided group automorphisms $\theta_V, \theta_v, \theta_\nu$. Evaluating the corresponding elements of our background quantum group $\tilde{H}$, etc., against the quantum matrix transformation of the x_i gives at once

$$\theta_V(x_i) = x_a V^a{}_i, \qquad \theta_{\mathrm{U}}(x_i) = x_a \mathrm{U}^a{}_i, \qquad \theta_\nu(x_i) = \lambda_\nu x_i,$$
$$V^i{}_j = \tilde{R}^i{}_a{}^a{}_j, \qquad \mathrm{U}^i{}_j = \tilde{R}^a{}_j{}^i{}_a, \qquad (\mathrm{UV})^{1/2} = \lambda_\nu\, \mathrm{id}, \tag{61}$$

where $\tilde{R} = ((R^{t_2})^{-1})^{t_2}$ is the second inverse (here t_2 denotes transposition in the second matrix factor of $M_n \otimes M_n$.) The action of the dilaton with its contribution λ^{ξ^2} to v, u and ν cancels the quantum group normalisation factor otherwise appearing in these formulae. The θ_ν applies in the case that the background quantum group is ribbon and the matrix representation is irreducible, which is typical in examples. The extension of the $\theta_V, \theta_{\mathrm{U}}, \theta_\nu$ is as algebra homomorphisms.

Proposition 5.3. *For the examples $\bar{M}_q(2)\rtimes U_q(su_2)\widetilde{\otimes} U_q(su_2)$ and $BM_q(2)\rtimes U_q(su_2)$ $\widetilde{\bowtie}$ $U_q(su_2)$ of four-dimensional q-Euclidean and q-Minkowski–Poincaré groups, we have*

$$V = \begin{pmatrix} q^{-4} & 0 & 0 & 0 \\ 0 & q^{-6} & 0 & 0 \\ 0 & 0 & q^{-2} & 0 \\ 0 & 0 & 0 & q^{-4} \end{pmatrix},$$
$$\mathrm{U} = \begin{pmatrix} q^{-4} & 0 & 0 & 0 \\ 0 & q^{-2} & 0 & 0 \\ 0 & 0 & q^{-6} & 0 \\ 0 & 0 & 0 & q^{-4} \end{pmatrix}, \qquad \lambda_\nu = q^{-4}.$$

Moreover, on the q-Euclidean and q-Minkowski space–time co-ordinates we have

$$\theta_V((\boldsymbol{x}_1 \cdots \boldsymbol{x}_n)^\star) = (\boldsymbol{x}_1 \cdots \boldsymbol{x}_n)^\star \lambda_\nu^n.$$

Proof. The first part is best computed from (61) using the appropriate $\boldsymbol{R}$ from [26,28], but can also be done directly from the v, u, ν elements in each copy of $U_q(su_2)$. Moreover, the $\star$ structure on the q-space–time co-cordinates in these two cases comes out from (59) as

$$\begin{pmatrix} a^\star & b^\star \\ c^\star & d^\star \end{pmatrix} = \begin{pmatrix} d & -qc \\ -q^{-1}b & a \end{pmatrix}, \qquad \begin{pmatrix} a^\star & b^\star \\ c^\star & d^\star \end{pmatrix} = \begin{pmatrix} a & q^2 c \\ q^{-2}b & d \end{pmatrix} \tag{62}$$

for the algebras $\bar{M}_q(2)$ and $BM_q(2)$, respectively. From this we see at once that $\theta_V(x_i^\star) = x_i^\star \lambda_\nu$ where $*$ is the standard $*$-structure on the space–time co-ordinates (obeying the unitarity condition (19)) for the two cases. We denote the space–time co-ordinates in both cases by x_i and the general form of $*$ is [8]

$$x_i^* = \begin{cases} x_a \eta^{ai}, & \text{real type I,} \\ x_{\bar{i}}, & \text{real type II.} \end{cases}$$

Our background quantum groups in these examples are ribbon and $c^\star = \sigma \triangleright c^*$ where $\sigma = u\nu^{-1}$ is Drinfeld's group-like element [38] computed from that of $U_q(su_2)$. So we can also take the point of view leading to (52) for these examples. □

We see that $\theta_V \circ \star$ is not an involution, but it is very close to one, differing only by multiples in each degree. There is a similar form for the $SO_q(n)$-covariant $\mathcal{R}_q^n$ spaces in [32] regarded as linear braided groups. Finally, we need for our constructions an invariant integral. There are two problems here, both of which can be addressed. The first is that we cannot expect polynomials in the co-ordinate generators x_i on our q-deformed linear spaces to be integrable; this can be handled by defining directly a Gaussian-weighted integral instead [24]. The second problem is that we cannot expect the integral to be invariant under the dilaton part λ^ξ of the inhomogeneous quantum group. This second problem is dealt with by slightly generalising our Lemma 4.7 as follows: suppose that ϕ is a linear functional $C \to \mathbb{C}$ which is invariant under some subalgebra of the quasi-$*$ Hopf algebra, and that the coproduct $\Delta h = h_{(1)} \otimes h_{(2)}$ of a given element of the latter can be written with the $h_{(2)}$ parts lying in the subalgebra. Then we can still conclude $(h^* \bar{\triangleright} b, c)^\theta_\phi = (b, h \triangleright c)^\theta_\phi$ as before. This is evident from (44) where we wrote the required step needed in Lemmas 4.6 and 4.7 quite explicitly. So we consider now a map $\int$ on $V^\vee(R', R)$ which is invariant under translation with respect to $\bar\partial$ and under the background quantum group without the dilaton. Hence it is invariant under the subalgebra of the inhomogeneous quantum group without the dilaton. From the form of the coproduct in Proposition 3.1 we see that we have the right form for all the generators except the dilaton λ^ξ. So the the fundamental and conjugate fundamental representations are indeed mutually adjoint as regrads the actions of $\boldsymbol{p}, \boldsymbol{l}^\pm$ (but not of λ^ξ) with respect to

$$(b, c) = \int (\lambda_\nu^\xi \triangleright b^*) c.$$

Here the action of λ_ν^ξ is multiplication of x_i by λ_ν in the setting of Proposition 5.3 and similar examples.

More precisely, the appropriately covariant integration functional on general braided linear spaces has been constructed in [24] in a Gaussian-weighted form. We need to give a right-invariant version appropriate to $\bar\partial$ rather than ∂ as given there. Briefly, suppose formally that there is a Gaussian $\bar g$ solving the equation $\bar\partial^i \bar g = -x_a \eta^{ai} \bar g$ as a power series. With the restrictions on R, η in [24, Section V.1] it takes the form of a q-exponential $\bar g = e_{\lambda^2}^{-(1+q^2)^{-1}\boldsymbol{x}\cdot\boldsymbol{x}}$ which is central and invariant under $\boldsymbol{l}^\pm$ and $*$. We do not need its precise form explicitly, however. Instead, we define directly a linear functional $\mathcal{Z} : C \to \mathbb{C}$ which plays the role of the ratio $\mathcal{Z}(f(\boldsymbol{x})) = \int f(\boldsymbol{x})\bar g / \int \bar g$. We regard the left-hand side as a definition of the right-hand side. The former, in turn, is defined directly in terms of the R-matrix R by means of induction as cf. [24]

$$\mathcal{Z}[1] = 1, \qquad \mathcal{Z}[x_i] = 0, \qquad \mathcal{Z}[x_i x_j] = \lambda^{-2} \eta_{ab} R^{-1}{}^a{}_j{}^b{}_i,$$

$$\mathcal{Z}[x_{i_1}\cdots x_{i_m}] = \sum_{r=0}^{m-2} \mathcal{Z}[x_{i_1}\cdots x_{i_r} x_{a_{r+3}}\cdots x_{a_m}] \\ \times \mathcal{Z}[x_{i_{r+1}} x_{a_{r+2}}][r+2, m;\, R_{21}^{-1}]^{a_{r+2}\cdots a_m}_{i_{r+2}\cdots i_m} \lambda^{-2(m-2-r)}. \tag{63}$$

We refer to [24] for the detailed derivation (for the left-handed case appropriate to ∂). For the example of $SO_q(n)$-covariant quantum planes such Gaussian-weighted integrations are known by other more explicit calculations as well [63]. We conclude in particular that ∂, $\bar\partial$ are mutually adjoint with respect to the sesqulinear form defined implicitly by

$$\frac{(b(\boldsymbol{x}), c(\boldsymbol{x})\bar g)}{(1, \bar g)} = \frac{\int \lambda_\nu^{|b|} b^* c\bar g}{\int \bar g} = \lambda_\nu^{|b|}\mathcal{Z}(b^* c), \tag{64}$$

where b, c are multinomials in x_i and $|\ \ |$ is the degree. More precisely, we take the right-hand side directly as a definition of a Gaussian-weighted sesquilinear form $\mathcal{Z}(b, c) = \lambda_\nu^{|b|}\mathcal{Z}(b^* c)$ even when $\int$ and $\bar g$ are not defined. The adjointness property then becomes

$$\mathcal{Z}((\boldsymbol{l}^{\pm})^* \triangleright b, c) = \mathcal{Z}(b, \boldsymbol{l}^{\pm} \triangleright c), \\ \mathcal{Z}((\partial^i)^* b, c) = -\mathcal{Z}(b, \bar\partial^i c) + \mathcal{Z}(b, \cdot\Psi(x_a \otimes c))\eta^{ai}\lambda^{2|c|}, \tag{65}$$

where $\cdot\Psi(x_{i_1}\otimes x_{i_2}\cdots x_{i_m}) = x_{a_1}\cdots x_{a_m}[1, m;\, R_{21}]^{a_m\cdots a_1}_{i_m\cdots i_1}$ from the standard braiding in the covector algebra [22], and $\partial^{i*} = \eta_{ia}\partial^a$, etc. is as for p^{i*} in (59), depending on the case. The two terms on the right come immediately from the computation of $\bar\partial(c\bar g)$ using the braided-Leibniz rule followed by the equation for the Gaussian. Equivalently, we use the Poincaré algebra coproduct (26) and the action of $\lambda^\xi, \boldsymbol{l}^-$ in Proposition 5.1. Finally, once (65) is obtained it may be verified directly for our q-Minkowski and q-Euclidean examples (at least to low order) on multinomials b, c even when $\int, \bar g$ are not given. A formal inductive proof is rather long and will be considered elsewhere. Non-degeneracy is also clear for our standard examples (and for generic q) since it holds for $q = 1$. Finally, our standard examples are unimodular which, combined with [8, Section 4], tells us that $\mathcal{Z} = \overline{\mathcal{Z}((\ \ )^*)}$, as one may verify directly for low order. This means that our sesquilinear form is conjugation-symmetric in the deformed sense

$$\overline{\mathcal{Z}(c, b)} = (\lambda_\nu^{|c|}/\lambda_\nu^{|b|})\mathcal{Z}(b, c). \tag{66}$$

One can also leave out λ_ν in the definition of $\mathcal{Z}(\ ,\ )$ and have the more standard conjugation-symmetry, but at the price of a spurious factor λ_ν^{-1} on the left-hand side of the adjointness property of ∂ in (65). This adjointness, combined with Corollary 5.2 is the sense in which the differential representation of the q-Poincaré algebra or inhomogeneous quantum group is 'unitary' in our braided approach. The sesquilinear form $\mathcal{Z}(\ ,\ )$ appears to be the appropriate starting point for q-quantum mechanics and q-scaler field theory in this approach. It remains to develop suitable tools (such as a braided version of Wick's theorem) for the computation of these Gaussian-weighted integrals and 'braided L^2 inner products' in a more closed form. It also remains to consider the appropriate formulation of completions, domains of operators, etc. for the corresponding functional analysis. This is a direction for further work.

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